THE FIRST-ORDER COHOMOLOGY GROUP OF SOME COMMUTATIVE SEMIGROUP ALGEBRAS

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ABSTRACT

In this paper we calculate the first-order cohomology group $H^1(ℓ^1(S), ℓ^∞(S))$, where $S$ is a commutative, 0-cancellative, nil*-semigroup.

KEYWORDS

semigroup, semigroup algebra, Cohomology.

1. INTRODUCTION

In [1], Bowling and Duncan investigated the first-order cohomology group $H^1(ℓ^1(S), ℓ^∞(S))$ and $H^1(ℓ^1(S), ℓ^1(S))$ for some classes of discrete semigroups, such as Clifford semigroups, Rees semigroups, and bicyclic semigroups. They also studied the cyclic cohomology in these cases. For a Banach algebra $A$ and a Banach $A$-bimodule $X$, it was shown that it is often possible to compute $H^1(A,X)$, where $X$ is $A$, or $X$ is $A^*$, with their bimodule products. For example, in the case of bicyclic semigroups $S$, it was proved that $H^1(ℓ^1(S), ℓ^∞(S))$ is isomorphic to $ℓ^∞(N)$.

In our result we shall establish a relationship between the first-order cohomology group $H^1(ℓ^1(S), ℓ^∞(S))$, where the semigroup $S$ is a commutative, 0-cancellative, nil*-semigroup, and the direct sum $⊕_{x≠0∈S} V_S^*(x) ⊕_{C∈C} W_C^*$ for non-zero elements $x$ of $S$, where the direct sum is in the sense of $ℓ^∞$.

2. Preliminaries

Let $S$ be a semigroup. An element $e ∈ S$ is an identity if $es = se = s$ $(s ∈ S)$. A semigroup with an identity is a unital semigroup.

Suppose that $S$ does not have an identity. Then we choose $e ∉ S$, and set $S^# = S ∪ \{e\}$ with $es = se = s$ $(s ∈ S)$ and $e^2 = e$. Then $S^#$ is a semigroup, called the unitization of $S$.

Let $S$ be a semigroup, and $x, y ∈ S$. Then $y|x$ means that $x ∈ yS^#$. A zero of $S$ is an element $o ∈ S$ with $os = so = o^2 = o$ $(s ∈ S)$.

Definition 2.1 Let $S$ be a unital, commutative semigroup. Then, for $x ∈ S$, we define

$$M_x = \{(y, z) ∈ S × S; yz = x\} \quad \text{and} \quad V_S(x) = \{y ∈ S; x ∈ yS\}. \quad (2.1)$$

We call $V_S(x)$ the set of divisors of $x$.

Note that $a, b ∈ V_S(x)$ whenever $ab ∈ V_S(x)$, and also $x ∈ V_S(x)$ because $S$ has an identity.

The following notion (in a different, additive notation) is given in [2], §4.

Definition 2.2 Let $S$ be a unital commutative semigroup. For each $x ∈ S$, we define the space $V_S^*(x)$ to consist of the bounded functions $g: V_S(x) → ℂ$ satisfying the logarithmic condition
\[ g(ab) = g(a) + g(b) \quad (2.2) \]
whenever \( a, b \in S \) and \( ab \in V_\delta(x) \).

Clearly \( V_\delta^*(x) \) is a linear space containing the zero function.

Note that for \( V_\delta(e) = \{ e \} \) and that \( V_\delta^*(e) = \{ 0 \} \). Also, in the case where \( S \) has a zero \( o \), \( V_\delta(o) = S \) and \( V_\delta^*(o) = \{ 0 \} \).

**Example 2.3** Take \( S = \mathbb{R}^+ \times \mathbb{R}^+ \) with normal addition. Then \( S \) is a unital, commutative semigroup, and \( \dim V_\delta^*(x) \geq 2 \) for some non-zero \( x \in S \).

Take the non-zero element \( x = (1,1) \) in \( S \). So that we have \( V_\delta(x) = [0,1] \times [0,1] \). For \((r,s) \in V_\delta(x)\), define the functions \( g_1,g_2:V_\delta(x) \to \mathbb{C} \) by \( g_1(r,s) = r \) and \( g_2(r,s) = s \). Then \( g_1, g_2 \in V_\delta^*(x) \) and, since \( g_1 \) and \( g_2 \) are linearly independent, \( \dim V_\delta^*(x) \geq 2 \).

**Proposition 2.4** Let \( S \) be a unital, commutative semigroup and suppose that \( x \in S \) is a non-zero element with \( \dim V_\delta^*(x) \geq 2 \). Then there exists a non-zero \( g \in V_\delta^*(x) \) with \( g(x) = 0 \).

**Proof** Let \( g_1 \) and \( g_2 \) be linearly independent functions in \( V_\delta^*(x) \). If \( g_1(x) = 0 \), then take \( g = g_1 \). Otherwise consider

\[ g = g_2 - \frac{g_2(x)}{g_1(x)} \cdot g_1. \]

Then \( g \in V_\delta^*(x) \) and \( g(x) = 0 \). Thus the proposition is proved.

Suppose that \( S \) is a commutative, 0-cancellative semigroup, that \( r \in S \backslash \{ o \} \), and that \( x \in V_\delta(r) \). Then there exists a unique element \( y \in V_\delta(r) \) such that \( r = xy \).

Note that for \( r = o \), an element \( y \) such that \( xy = o \) is not necessarily unique.

**Definition 2.5** Let \( S \) be a commutative, 0-cancellative semigroup. For each non-zero element \( r \in S \), the unique element \( y \in V_\delta(r) \) of \( x \in V_\delta(r) \) such that \( xy = r \) is called \( u(x) \).

The following is a small modification of the set \( M_x \) that we defined in Definition 2.1.

**Definition 2.6** Let \( S \) be a unital, commutative semigroup with zero \( o \). We define the set

\[ M_0^- = \{ (a,b) \in S \times S : a \neq o, b \neq o \quad \text{and} \quad ab = o \} \].

We define an equivalence relation \( \sim \) on the set \( M_0^- \). It is the equivalence relation generated by the relations:

1. \((a_1a_2,c) \sim (a_1,a_2c)\) for all \( a_1,a_2,c \in S \) with \( a_1a_2c = o \) and \( a_1a_2 \neq o, a_2c \neq o \); and
2. \((a,b) \sim (b,a)\) for \( (a,b) \in M_0^- \).

It is possible that \( M_0^- = \emptyset \). Indeed, take \( S = \{ o, e \} \). Then \( M_0^- = \emptyset \).

**Definition 2.7** Let \( S \) be a unital, commutative semigroup with zero \( o \) and identity \( e \). Let \( C \subseteq M_0^- \) be an equivalence class, and let \( \varphi: C \to \mathbb{C} \) be a bounded function. Then we define the set \( \tilde{C} = C \cup (S \times \{ o \}) \cup (\{ o \} \times S) \), and extend the function \( \varphi \) to a function \( \tilde{\varphi}: \tilde{C} \to \mathbb{C} \) satisfying

\[ \tilde{\varphi}(o,a) = \tilde{\varphi}(a,o) = 0 \quad (a \in S). \quad (2.3) \]

Then the function \( \varphi \) is sensible if we have

\[ \tilde{\varphi}(a,bc) + \tilde{\varphi}(b,ac) = \tilde{\varphi}(ab,c) \quad (2.4) \]
whenever \( (a,bc) \in \tilde{C} \) or \( (b,ac) \in \tilde{C} \) or \( (ab,c) \in \tilde{C} \).
Note that \( \tilde{\varphi} \) is defined at all three pairs \((a, bc), (b, ac), (ab, c)\) whenever any one of these pairs is in the equivalence class \( \tilde{C} \). E.g. if \((a, bc) \in \tilde{C} \), then either \((ac, b) \in \tilde{C} \), and hence \((b, ac) \in \tilde{C} \), or \((ac, b) = (a, b)\).

Note that (2.4) implies that
\[
\varphi(a, b) + \varphi(b, a) = \tilde{\varphi}(ab, e) = \tilde{\varphi}(a, e) = 0 \quad ((a, b) \in C). \quad (2.5)
\]
The sensible functions on an equivalence class \( C \) form a linear space, which we call \( \mathcal{W}_C^* \), and they have the uniform norm
\[
\|\varphi\|_\infty = \sup_{(a, b) \in C} |\varphi(a, b)|.
\]
The space \( \mathcal{W}_C^* \) is a closed linear subspace of the Banach space \( \ell^\infty(C) \).

We denote the collection of all such equivalence classes by \( \mathcal{C} \).

**Definition 2.8** Let \( S \) be a semigroup. Then the convolution product of two elements \( f \) and \( g \) in the Banach space \( \ell^1(S) \) is defined by the formula:
\[
f \ast g = (\sum_{s \in S} \alpha_s \delta_s) \ast (\sum_{t \in S} \beta_t \delta_t) = \sum \{ \sum_{s = t \in S} \alpha_s \beta_t \delta_t \}.
\]
The inner sum will vanish if there are no \( s \) and \( t \) in \( S \) such that \( st = r \). Clearly, \( (\ell^1(S)) \ast \) is a Banach algebra; it is called the semigroup algebra of \( S \).

The dual space of \( \mathcal{A} = \ell^1(S) \) is \( \mathcal{A}^* = \ell^\infty(S) \), where
\[
\ell^\infty(S) = \left\{ f: S \to \mathbb{C} : \|f\| = \sup_{s \in S} |f(s)| < \infty \right\},
\]
with the duality given by:
\[
\langle f, \lambda \rangle = \sum_{s \in S} f(s) \lambda(s) \quad (f \in \ell^1(S), \lambda \in \ell^\infty(S)).
\]

### 3. The main result

In our result we shall establish a relationship between the first-order cohomology group \( \mathcal{H}^1(\ell^1(S), \ell^\infty(S)) \), where the semigroup \( S \) is commutative, 0-cancellative, \( nnil^# \)-semigroup, and the direct sum \( \bigoplus_{x \neq 0 \in S} V_S^* (x) \bigoplus_{C \in C} \mathcal{W}_C^* \) for non-zero elements \( x \) of \( S \), where the direct sum is in the sense of \( \ell^\infty \).

**Theorem 3.1** Let \( S \) be a commutative, 0-cancellative, \( nnil^# \)-semigroup. Then
\[
\mathcal{H}^1(\ell^1(S), \ell^\infty(S)) \cong \bigoplus_{x \neq 0 \in S} V_S^* (x) \bigoplus_{C \in C} \mathcal{W}_C^*,
\]
where the sum is an \( \ell^\infty \)-direct sum.

**Proof** We define an isomorphism
\[
\Theta: \bigoplus_{x \neq 0 \in S} V_S^* (x) \bigoplus_{C \in C} \mathcal{W}_C^* \to \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*)
\]
as follows: Given bounded families \( (g_x) \in \bigoplus_{x \neq 0 \in S} V_S^* (x) \) and \( (\varphi_C) \in \bigoplus_{C \in C} \mathcal{W}_C^* \), so that \( ((g_x), (\varphi_C)) \) belongs to the \( \ell^\infty \)-direct sum, we define \( \gamma: S \times S \to \mathbb{C} \) such that
We now consider the map
\[ \gamma(s, t) = \begin{cases} 
0 & \text{if } s = 0 \text{ or } t = 0, \\
g_{st}(s) & \text{if } st \neq 0, \\
\varphi_c(s, t) & \text{if } st = 0 \text{ and } (s, t) \in \mathcal{C} 
\end{cases} \] (3.1)

Now set \( \mathcal{A} = \ell^1(S) \), and define a map \( D: \mathcal{A} \to \mathcal{A}^* \) by the relation:
\[ \langle \delta, D(\delta) \rangle = \gamma(s, t) \quad (s, t \in S). \] (3.2)

The map \( D \) extends to a linear map, and \( D \) is bounded because the functions \( \varphi_c \) and \( g_x \) are uniformly bounded.

Note that \( D(\delta_o) = 0 = \langle \delta_o, D(\delta) \rangle \) for each \( s \in S \).

We claim (essentially following [2, Proposition 4.2]), that \( D \) is a derivation.

To prove this, take the elements \( u, v, t \in S \). We shall show that
\[ \langle \delta, D(\delta) \rangle = \langle \delta, uD(\delta) + D(\delta) v \rangle = 0. \]

That is we shall show that \( \gamma(uv, t) = \gamma(u, vt) + \gamma(v, ut) \).

In the case where \( uv = o \), we have to discuss the following two cases:

**Case 1:** If at least two of \( u, v \) and \( t \) are zero, so that \( uv, vt \) and \( ut \) are zero, then by using (3.1) we have \( \gamma(uv, t) = \gamma(u, vt) + \gamma(v, ut) = 0 \).

**Case 2:** If at most one of \( u, v \) and \( t \) are zero, then we have to look at two possibilities.

Firstly, if each of the pairs \( (uv, t), (ut, v) \) and \( (vt, u) \) contains a zero element, we still have
\[ \gamma(uv, t) = \gamma(u, vt) = \gamma(v, tu) = 0. \]

Secondly, if at least one of the pairs \( (uv, t), (ut, v) \), and \( (vt, u) \) has both elements non-zero, say \( uv \neq o \neq t \), then \( (uv, t) \) must belong to an equivalence class \( C \) if \( ut \neq o \), then \( (v, ut) \in C \) and, if \( vt \neq o \), then \( (u, vt) \in C \), so that by using (2.4), we have
\[ \gamma(uv, t) - \gamma(u, vt) - \gamma(v, ut) = \varphi_c(uv, t) - \varphi_c(u, vt) - \varphi_c(v, ut) = 0. \]

In the case where \( uv \neq o \), we have
\[ \langle \delta, D(\delta) \rangle = \gamma(uv, t) = g_{uv}(uv) = g_{uv}(u) + g_{uv}(v) = \gamma(u, vt) + \gamma(v, ut) \]
\[ = \langle \delta u, D(\delta) \rangle + \langle \delta v, D(\delta) \rangle = \langle \delta s, \delta t \rangle D(\delta) + D(\delta) \cdot (\delta s, \delta t). \]

Thus \( D \) is a bounded derivation.

The derivation \( D \) depends on our choice of the function \( g_x \in V^*_x(x) \) and the function \( \varphi_c \in W^*_c \). Given bounded families \( g_x \in V^*_x(x) \quad (x \in S) \) and \( \varphi_c \in W^*_c \quad (C \in \mathcal{C}) \), then we have an element denoted by \( D[(g_x)_{x \in S}, (\varphi_c)_{c \in \mathcal{C}}] \in \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) \).

We now consider the map
\[ \Theta: \bigoplus_{x \neq o \in S} V^*_x(x) \bigoplus_{c \in \mathcal{C}} W^*_c \to \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) \]
such that
\[ \Theta((g_x), (\varphi_c)) = D[(g_x)_{x \in S}, (\varphi_c)_{c \in \mathcal{C}}]. \]
Clearly $\Theta$ is linear.

Suppose that $D[(g_s)_{s \in S}, (\varphi_C)_{C \in C}] = 0$. Then $\langle \delta_t, D(\delta_s) \rangle = 0$ for all $s, t \in S$, and so $g_{st}(s) = \varphi_C(s, s_t) = 0$ whenever $s = s_1 s_2, t \in S$ and $st \neq o, s_1 s_2 t = o$. This shows that $\Theta$ is injective.

Finally, to see that $\Theta$ is surjective, suppose that $D_0 : A \to A^*$ is a derivation, and then define $\gamma(s, t) = \langle \delta_t, D_0(\delta_s) \rangle$ for all $s, t = o$ in $S$.

We claim that, for $x \neq o$, the function $\gamma$ is of form $g_x(s)$ for some $g_x \in V_2(x)$ when restricted to the set $M_\gamma = \{(s, t) : st = x\}$. We also claim that for each $C \in C$, the function $\gamma_C = \gamma(C)$ is an element of $W_C^*$; and in fact $\gamma(s, t) = 0$ if $s = o$ or $t = o$. Then clearly that $g_x$ and $\varphi_C$ must be uniformly bounded otherwise $D_0$ is not a bounded derivation, so that we have

$$D_0 = D[(g_s)_{s \in S}, (\varphi_C)_{C \in C}] .$$

Since $D(\delta_o) = 0$, whenever $\gamma(o, t) = 0$ for all $t \in S$. Also for $s \in S$ we have that

$$\gamma(s, o) = \langle \delta_o, D(\delta_s) \rangle = \langle \delta_s, \delta_o D(\delta_o) \rangle = \langle \delta_1, \delta_o D(\delta_2) - \delta_o D(\delta_2) \rangle = 0 ,$$

so that $\gamma(s, t) = 0$ whenever $s = o$ or $t = o$.

Now restrict $\gamma$ to $M_\gamma$ for $x \neq o$. We claim that there exists $g_x \in V_2(x)$ with $\gamma(s, t) = g_x(s)$. We do not give proof because this is essentially a repeat of a previous proof.

Restrict $\gamma$ to $\tilde{C} \in C$. We claim that $\gamma(\tilde{C})$ is an element of $W_C^*$, and so that $\gamma$ is sensible.

To prove our claim, we shall see that

$$\tilde{\gamma}(a b, c) = \tilde{\gamma}(a, b c) + \tilde{\gamma}(b, a c) \quad (3.3)$$

whenever $(a b, c) \in \tilde{C}$ or $(a, b c) \in \tilde{C}$ or $(c, a b) \in \tilde{C}$.

In fact $\tilde{\gamma}(u, v) = \langle \delta_v, D(\delta_u) \rangle$, and so, by using (3.3), for $a, b, c \in S$, we have

$$\tilde{\gamma}(a b, c) = \langle \delta_c, D(\delta_{a c}) \rangle = \langle \delta_c, \delta_b D(\delta_a) + \delta_a D(\delta_b) \rangle$$

Thus the theorem is proved.

**Corollary 3.2** Let $S$ be a commutative, 0-cancellative, nil #-semigroup. Then

$$\dim H^1(\ell_1(S), \ell_0^\infty(S)) = \sum_{x \neq o \in S} \dim V_2^*(x) + \sum_{C \in C} \dim W_C^* .$$

**REFERENCES**