# THE FIRST-ORDER COHOMOLOGY GROUP OF SOME COMMUTATIVE SEMIGROUP ALGEBRAS

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***ABSTRACT***

*In this paper we calculate the first-order cohomology group $\mathcal{H}^1(\ell^1(S), \ell^\infty(S))$, where $S$ is a commutative, 0-cancellative, $nil^\#$-semigroup.*

***KEYWORDS***

*semigroup, semigroup algebra, Cohomology.*

## 1. INTRODUCTION

In [1], Bowling and Duncan investigated the first-order cohomology group $\mathcal{H}^1(\ell^1(S), \ell^\infty(S))$ and $\mathcal{H}^1(\ell^1(S), \ell^1(S))$ for some classes of discrete semigroups, such as Clifford semigroups, Rees semigroups, and bicyclic semigroups. They also studied the cyclic cohomology in these cases. For a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule $X$, it was shown that it is often possible to compute $\mathcal{H}^1(\mathcal{A}, X)$, where $X$ is $\mathcal{A}$, or $X$ is $\mathcal{A}^*$, with their bimodule products. For example, in the case of bicyclic semigroups $S$, it was proved that $\mathcal{H}^1(\ell^1(S), \ell^\infty(S))$ is isomorphic to $\ell^\infty(\mathbb{N})$.
In our result we shall establish a relationship between the first-order cohomology group $\mathcal{H}^1(\ell^1(S), \ell^\infty(S))$, where the semigroup $S$ is a commutative, 0-cancellative, $nil^\#$-semigroup, and the direct sum $\bigoplus_{x \neq o \in S} V_S^*(x) \bigoplus_{C \in \mathcal{C}} W_C^*$ for non-zero elements $x$ of $S$, where the direct sum is in the sense of $\ell^\infty$.

## 2. Preliminaries

Let $S$ be a semigroup. An element $e \in S$ is an *identity* if $es = se = s \quad (s \in S)$. A semigroup with an identity is a *unital semigroup*.

Suppose that $S$ does not have an identity. Then we choose $e \notin S$, and set $S^\# = S \cup \{e\}$ with $es = se = s \quad (s \in S)$ and $e^2 = e$. Then $S^\#$ is a semigroup, called the *unitization* of $S$.

Let $S$ be a semigroup, and $x, y \in S$. Then $y|x$ means that $x \in yS^\#$.

A *zero* of $S$ is an element $o \in S$ with $os = so = o^2 = o \quad (s \in S)$.

**Definition 2.1** *Let $S$ be a unital, commutative semigroup. Then, for $x \in S$, we define*

$$M_x = \{(y, z) \in S \times S : yz = x\} \quad and \quad V_S(x) = \{y \in S : x \in yS\}. \tag{2.1}$$

We call $V_S(x)$ the *set of divisors* of $x$.

Note that $a, b \in V_S(x)$ whenever $ab \in V_S(x)$, and also $x \in V_S(x)$ because $S$ has an identity.

The following notion (in a different, additive notation) is given in [2], §4.

**Definition 2.2** *Let $S$ be a unital commutative semigroup. For each $x \in S$, we define the space $V_S^*(x)$ to consist of the bounded functions $g: V_S(x) \to \mathbb{C}$ satisfying the logarithmic condition*

$$g(ab) = g(a) + g(b) \quad (2.2)$$

whenever $a, b \in S$ and $ab \in V_S(x)$ .

Clearly $V_S^*(x)$ is a linear space containing the zero function.

Note that $V_S(e) = \{e\}$ and that $V_S^*(e) = \{0\}$ . Also, in the case where $S$ has a zero $o$ , $V_S(o) = S$ and $V_S^*(o) = \{0\}$ .

**Example 2.3** *Take $S = \mathbb{R}^+ \times \mathbb{R}^+$ with normal addition. Then $S$ is a unital, commutative semigroup, and $dimV_S^*(x) \geq 2$ for some non-zero $x \in S$ .*

Take the non-zero element $x = (1,1)$ in $S$. So that we have $V_S(x) = [0,1] \times [0,1]$ . For $(r, s) \in V_S(x)$, define the functions $g_1, g_2: V_S(x) \to \mathbb{C}$ by $g_1(r, s) = r$ and $g_2(r, s) = s$ . Then $g_1, g_2 \in V_S^*(x)$ and, since $g_1$ and $g_2$ are linearly independent, $\dim V_S^*(x) \geq 2$ . ∎

**Proposition 2.4** *Let $S$ be a unital, commutative semigroup and suppose that $x \in S$ is a non-zero element with $dimV_S^*(x) \geq 2$ . Then there exists a non-zero $g \in V_S^*(x)$ with $g(x) = 0$ .*

**Proof** Let $g_1$ and $g_2$ be linearly independent functions in $V_S^*(x)$. If $g_1(x) = 0$, then take $g = g_1$ . Otherwise consider

$$g = g_2 - \frac{g_2(x)}{g_1(x)} \cdot g_1 .$$

Then $g \in V_S^*(x)$ and $g(x) = 0$ . Thus the proposition is proved. ∎

Suppose that $S$ is a commutative, 0-cancellative semigroup, that $r \in S \setminus \{o\}$, and that $x \in V_S(r)$ . Then there exists a unique element $y \in V_S(r)$ such that $r = xy$ .

Note that for $r = o$, an element $y$ such that $xy = o$ is not necessarily unique.

**Definition 2.5** *Let $S$ be a commutative, 0-cancellative semigroup. For each non-zero element $r \in S$, the unique element $y \in V_S(r)$ of $x \in V_S(r)$ such that $xy = r$ is called $u(x)$ .*

The following is a small modification of the set $M_x$ that we defined in Definition 2.1 .

**Definition 2.6** *Let $S$ be a unital, commutative semigroup with zero $o$ . We define the set*

$$M_o^- = \{(a, b) \in S \times S: a \neq o, b \neq o \quad and \quad ab = o\} .$$

We define an equivalence relation $\sim$ on the set $M_o^-$ . It is the equivalence relation generated by the relations:

1. $(a_1 a_2, c) \sim (a_1, a_2 c)$ for all $a_1, a_2, c \in S$ with $a_1 a_2 c = o$ and $a_1 a_2 \neq o$ , $a_2 c \neq o$ ; and
2. $(a, b) \sim (b, a)$ for $(a, b) \in M_o^-$ .

It is possible that $M_o^- = \emptyset$ . Indeed, take $S = \{o, e\}$ . Then $M_o^- = \emptyset$ .

**Definition 2.7** *Let $S$ be a unital, commutative semigroup with zero $o$ and identity $e$. Let $C \subset M_o^-$ be an equivalence class, and let $\varphi: C \to \mathbb{C}$ be a bounded function. Then we define the set $\tilde{C} = C \cup (S \times \{o\}) \cup (\{o\} \times S)$, and extend the function $\varphi$ to a function $\tilde{\varphi}: \tilde{C} \to \mathbb{C}$ satisfying*

$$\tilde{\varphi}(o, a) = \tilde{\varphi}(a, o) = 0 \quad (a \in S) . \qquad (2.3)$$

Then the function $\varphi$ is *sensible* if we have

$$\tilde{\varphi}(a, bc) + \tilde{\varphi}(b, ac) = \tilde{\varphi}(ab, c) \qquad (2.4)$$

whenever $(a, bc) \in \tilde{C}$ or $(b, ac) \in \tilde{C}$ or $(ab, c) \in \tilde{C}$.

Note that $\tilde{\varphi}$ is defined at all three pairs $(a, bc)$ , $(b, ac)$, $(ab, c)$ whenever any one of these pairs is in the equivalence class $\tilde{C}$. E.g. if $(a, bc) \in \tilde{C}$, then either $(ac, b) \in \tilde{C}$ , and hence $(b, ac) \in \tilde{C}$ , or $(ac, b) = (o, b)$.

Note that (2.4) implies that

$$\varphi(a, b) + \varphi(b, a) = \tilde{\varphi}(ab, e) = \tilde{\varphi}(o, e) = 0 \quad ((a, b) \in C) . \qquad (2.5)$$

The sensible functions on an equivalence class $C$ form a linear space, which we call $W_C^*$ , and they have the uniform norm

$$\|\varphi\|_\infty = \sup_{(a,b)\in C} |\varphi(a, b)| .$$

The space $W_C^*$ is a closed linear subspace of the Banach space $\ell^\infty(C)$ .

We denote the collection of all such equivalence classes by $\mathcal{C}$.

**Definition 2.8** *Let $S$ be a semigroup. Then the convolution product of two elements $f$ and $g$ in the Banach space $\ell^1(S)$ is defined by the formula:*

$$f * g = (\sum_{s\in S} \alpha_s\delta_s) * (\sum_{t\in S} \beta_t\delta_t) = \sum \{(\sum_{st=r\in S} \alpha_s\beta_t)\delta_r\} .$$

The inner sum will vanish if there are no $s$ and $t$ in $S$ such that $st = r$. Clearly, $(\ell^1(S),*)$ is a Banach algebra; it is called the *semigroup algebra* of $S$.

The *dual space* of $\mathcal{A} = \ell^1(S)$ is $\mathcal{A}^* = \ell^\infty(S)$, where

$$\ell^\infty(S) = \left\{f: S \to \mathbb{C}: \quad \|f\| = \sup_{s\in S}|f(s)| < \infty\right\},$$

with the duality given by:

$$\langle f, \lambda\rangle = \sum_{s\in S} f(s)\lambda(s) \quad (f \in \ell^1(S), \lambda \in \ell^\infty(S)) .$$

## 3. The main result

In our result we shall establish a relationship between the first-order cohomology group $\mathcal{H}^1(\ell^1(S), \ell^\infty(S))$ , where the semigroup $S$ is commutative, 0-cancellative, $nil^\#$-semigroup, and the direct sum $\bigoplus_{x\neq o\in S} V_S^*(x) \bigoplus_{C\in\mathcal{C}} W_C^*$ for non-zero elements $x$ of $S$, where the direct sum is in the sense of $\ell^\infty$.

**Theorem 3.1** *Let $S$ be a commutative, 0-cancellative, $nil^\#$-semigroup. Then*

$$\mathcal{H}^1(\ell^1(S), \ell^\infty(S)) \cong \bigoplus_{x\neq o\in S} V_S^*(x) \bigoplus_{C\in\mathcal{C}} W_C^* ,$$

where the sum is an $\ell^\infty$- direct sum.

**Proof** We define an isomorphism

$$\Theta: \bigoplus_{x\neq o\in S} V_S^*(x) \bigoplus_{C\in\mathcal{C}} W_C^* \to \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*)$$

as follows: Given bounded families $(g_x) \in \bigoplus_{x\neq o\in S} V_S^*(x)$ and $(\varphi_C) \in \bigoplus_{C\in\mathcal{C}} W_C^*$ , so that $\big((g_x), (\varphi_C)\big)$ belongs to the $\ell^\infty$- direct sum, we define $\gamma: S \times S \to \mathbb{C}$ such that

$$\gamma(s,t) = \begin{cases} 0 & if\ s = 0\ or\ t = 0, \\ g_{st}(s) & if\ st \neq 0, \\ \varphi_C(s,t) & if\ st = 0\ \ and\ (s,t) \in C \end{cases} \qquad (3.1)$$

Now set $\mathcal{A} = \ell^{\,1}(S)$ , and define a map $D: \mathcal{A} \to \mathcal{A}^*$ by the relation:

$$\langle \delta_t, D(\delta_s) \rangle = \gamma(s,t) \quad (s,t \in S)\,. \qquad (3.2)$$

The map $D$ extends to a linear map, and $D$ is bounded because the functions $\varphi_C$ and $g_x$ are uniformly bounded.

Note that $D(\delta_o) = 0 = \langle \delta_o, D(\delta_s) \rangle$ for each $s \in S$ .

We *claim* ( essentially following [2, Proposition 4.2]), that $D$ is a derivation.

To prove this, take the elements $u, v, t \in S$. We shall show that

$$\langle \delta_t, D(\delta_{uv}) \rangle = \langle \delta_t, uD(\delta_v) + D(\delta_u)v \rangle = 0\,.$$

That is we shall show that $\gamma(uv,t) = \gamma(u,vt) + \gamma(v,ut)$ .

In the case where $uvt = o$, we have to discuss the following two cases:

**Case 1:** If at least two of $u$, $v$ and $t$ are zero, so that $uv$, $vt$ and $ut$ are zero, then by using (3.1) we have $\gamma(uv,t) = \gamma(u,vt) = \gamma(v,ut) = 0$ .

**Case 2:** If at most one of $u$, $v$ and $t$ are zero, then we have to look at two possibilities.

Firstly, if each of the pairs $(uv,t)$, $(ut,v)$ and $(vt,u)$ contains a zero element, we still have

$$\gamma(ut,v) = \gamma(u,tv) = \gamma(v,tu) = 0\,.$$

Secondly, if at least one of the pairs $(uv,t)$, $(ut,v)$ , and $(vt,u)$ has both elements non-zero, say $uv \neq o \neq t$, then $(uv,t)$ must belong to an equivalence class $C$. If $ut \neq o$, then $(v,ut) \in C$ and, if $vt \neq o$, then $(u,vt) \in C$, so that by using (2.4), we have

$$\gamma(uv,t) - \gamma(u,vt) - \gamma(v,ut) = \tilde{\varphi}_C(uv,t) - \tilde{\varphi}_C(u,vt) - \tilde{\varphi}_C(v,ut) = 0\,.$$

In the case where $uvt \neq o$ , we have

$$\langle \delta_t, D(\delta_{uv}) \rangle = \gamma(uv,t) = g_{uvt}(uv) = g_{uvt}(u) + g_{uvt}(v) = \gamma(u,vt) + \gamma(v,ut)$$
$$= \langle \delta_{vt}, D(\delta_u) \rangle + \langle \delta_{ut}, D(\delta_v) \rangle = \langle \delta_t, \delta_v \cdot D(\delta_u) + \delta_u \cdot D(\delta_v) \rangle\,.$$

Thus $D$ is a bounded derivation.

The derivation $D$ depends on our choice of the function $g_x \in V_S^*(x)$ and the function $\varphi_C \in W_C^*$ . Given bounded families $g_x \in V_S^*(x)$ $(x \in S)$ and $\varphi_C \in W_C^*$ $(C \in \mathcal{C})$, then we have an element denoted by $D[(g_x)_{x\in S}, (\varphi_C)_{C\in\mathcal{C}}] \in \mathcal{H}^{\,1}(\mathcal{A},\mathcal{A}^*)$ .

We now consider the map

$$\Theta: \bigoplus_{x\neq o\in S} V_S^*(x) \bigoplus_{C\in\mathcal{C}} W_C^* \to \mathcal{H}^{\,1}(\mathcal{A},\mathcal{A}^*)$$

such that

$$\Theta\big((g_x),(\varphi_C)\big) = D[(g_x)_{x\in S}, (\varphi_C)_{C\in\mathcal{C}}]\,.$$

Clearly $\Theta$ is linear.

Suppose that $D[(g_x)_{x\in S},(\varphi_C)_{C\in\mathcal{C}}]=0$. Then $\langle\delta_t,D(\delta_s)\rangle=0$ for all $s,t\in S$, and so $g_{st}(s_1)=\varphi_C(s_1,s_2t)=0$ whenever $s=s_1s_2,t\in S$ and $st\neq o$, $s_1s_2t=o$. This shows that $\Theta$ is injective.

Finally, to see that $\Theta$ is surjective, suppose that $D_0:\mathcal{A}\to\mathcal{A}^*$ is a derivation, and then define $\gamma(s,t)=\langle\delta_t,D_0(\delta_s)\rangle$ for all $st\neq o$ in $S$.

We *claim* that, for $x\neq o$, the function $\gamma$ is of form $g_x(s)$ for some $g_x\in V_S^*(x)$ when restricted to the set $M_x=\{(s,t):st=x\}$. We also *claim* that for each $C\in\mathcal{C}$, the function $\gamma_C=\gamma|C$ is an element of $W_C^*$; and in fact $\gamma(s,t)=0$ if $s=o$ or $t=o$. Then clearly that $g_x$ and $\varphi_C$ must be uniformly bounded otherwise $D_0$ is not a bounded derivation, so that we have

$$D_0=D[(g_x)_{x\in S},(\gamma_C)_{C\in\mathcal{C}}].$$

Since $D(\delta_o)=0$, whenever $\gamma(o,t)=0$ for all $t\in S$. Also for $s\in S$ we have that

$$\gamma(s,o)=\langle\delta_o,D(\delta_s)\rangle=\langle\delta_1,\delta_oD(\delta_s)\rangle=\langle\delta_1,\delta_oD(\delta_o\delta_s)-\delta_sD(\delta_o)\rangle=0,$$

so that $\gamma(s,t)=0$ whenever $s=o$ or $t=o$.

Now restrict $\gamma$ to $M_x$ for $x\neq o$. We claim that there exists $g_x\in V_S^*$ with $\gamma(s,t)=g_x(s)$. We do not give proof because this is essentially a repeat of a previous proof.

Restrict $\gamma$ to $\tilde{C}\in\mathcal{C}$. We *claim* that $\gamma|\tilde{C}$ is an element of $W_C^*$, and so that $\gamma$ is sensible.

To prove our claim, we shall see that

$$\tilde{\gamma}(ab,c)=\tilde{\gamma}(a,bc)+\tilde{\gamma}(b,ac)\quad(3.3)$$

whenever $(ab,c)\in\tilde{C}$ or $(a,bc)\in\tilde{C}$ or $(c,ab)\in\tilde{C}$.

In fact $\tilde{\gamma}(u,v)=\langle\delta_v,D(\delta_u)\rangle$, and so, by using (3.3), for $a,b,c\in S$, we have

$$\tilde{\gamma}(ab,c)=\langle\delta_c,D(\delta_{ab})\rangle=\langle\delta_c,\delta_bD(\delta_a)+\delta_aD(\delta_a)\rangle$$
$$=\langle\delta_{bc},D(\delta_a)\rangle+\langle\delta_{ac},D(\delta_b)\rangle=\tilde{\gamma}(a,bc)+\tilde{\gamma}(b,ac).$$

Thus the theorem is proved. ■

**Corollary 3.2** *Let S be a commutative, 0-cancellative, nil $^{\#}$-semigroup. Then*

$$\dim\mathcal{H}^1(\ell^1(S),\ell^\infty(S))=\sum_{x\neq o\in S}\dim V_S^*(x)+\sum_{C\in\mathcal{C}}\dim W_C^*. \qquad ■$$

## REFERENCES

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