CYCLIC WEAKLY AMENABLE SEMIGROUP ALGEBRAS WHICH ARE NOT WEAKLY AMENABLE

Hussein Mohamed Ghlaio
Department of Mathematics, Faculty of Science , Misurata University, Misurata, Libya
H.Ghlaio@gmail.com

Article information

Key words
semigroup algebra, cyclic weakly amenable.

Received 26 February 2021, Accepted 22 March 2021, Available online 01 April 2021

Abstract

In this paper we shall construct some examples where the semigroup algebra $\ell^1(S)$ is cyclic weakly amenable but not weakly amenable ,where S is a commutative, 0-cancellative, $nil^{\#}$-semigroup.

I. INTRODUCTION

We follow [1] to recall some definitions and some preliminaries. Let $\mathcal{A}$ be a Banach algebra, and let X be a *Banach $\mathcal{A}$-bimodule*. A linear map $D: \mathcal{A} \to X$ is a *derivation* if it satisfies the equation:

$$D(ab) = a \cdot Db + Da \cdot b \quad (a, b \in \mathcal{A}) .$$

In this paper we shall only consider bounded derivations. Given $x \in X$ and define the map $\delta_x: \mathcal{A} \to X$ by the equation:

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These derivations are *inner* derivations.

Let X^* be the *dual space* of X. Then X^* is a Banach $\mathcal{A}$-bimodule with respect to the operations given by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle \quad and$$
$$\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (a \in \mathcal{A}, x \in X, \lambda \in X^*) .$$

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation D from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule X^* is inner, for each Banach $\mathcal{A}$-bimodule X.

A Banach algebra $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule over itself. Then a Banach algebra $\mathcal{A}$ is *weakly amenable* if every bounded derivation $D: \mathcal{A} \to \mathcal{A}^*$ is inner.

A linear map $T: \mathcal{A} \to \mathcal{A}^*$ is cyclic if $T(a_1)(a_0) = (-1)T(a_0)(a_1)$ for all $a_0, a_1 \in \mathcal{A}$; in other words, $\langle a_0, T(a_1) \rangle + \langle a_1, T(a_0) \rangle = 0 \quad (a_0, a_1 \in \mathcal{A})$.

In particular, $\langle a, T(a) \rangle = 0 \quad (a \in \mathcal{A})$.

The space of all bounded, cyclic derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*)$, and the set of all cyclic inner derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*)$. It can be seen that every inner derivation is cyclic, and so $\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*) = \mathcal{N}^1(\mathcal{A}, \mathcal{A}^*)$. The *first-order cyclic cohomology group* is defined by

$$\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) = \frac{\mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*)}{\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*)} = \mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*)/\mathcal{N}^1(\mathcal{A}, \mathcal{A}^*) .$$

A Banach algebra $\mathcal{A}$ is *cyclic weakly amenable* if $\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$.

It is obvious that

$\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) \subseteq \mathcal{H}^1(\mathcal{A}, \mathcal{A}^*)$, so each weakly amenable Banach algebra is cyclic weakly amenable.

Let S be a non-empty set, and let s be an element of S. The characteristic function of $\{s\}$ is denoted by δ_s. We define the *Banach space*

$$\ell^1(S) := \left\{ f : S \to \mathbb{C},\ \ f = \sum_{s\in S} \alpha_s \delta_s, \sum_{s\in S} |\alpha_s| < \infty \right\},$$

where $\|f\| = \sum_{s\in S} |\alpha_s| < \infty$.

The *dual space* of $\mathcal{A} = \ell^1(S)$ is $\mathcal{A}^* = \ell^\infty(S)$, where

$$\ell^\infty(S) = \left\{ f : S \to \mathbb{C},\ \ \|f\| = \sup_{s\in S} |f(s)| < \infty \right\},$$

with the duality given by:

$$\langle f, \lambda \rangle = \sum_{s\in S} f(s)\lambda(s) \quad (f \in \ell^1(S), \lambda \in \ell^\infty(S)).$$

Let S be a semigroup. Then the *convolution product* of two elements f and g in the Banach space $\ell^1(S)$ is defined by the formula:

$$f * g = \left(\sum_{s\in S} \alpha_s \delta_s \right) * \left(\sum_{t\in S} \beta_t \delta_t \right) = \sum \left\{ \left(\sum_{st=r\in S} \alpha_s \beta_t \right) \delta_r \right\}$$

The inner sum will vanish if there are no s and t such that $st = r$.

Clearly, $(\ell^1(S), *)$ is a Banach algebra; it is called the *semigroup algebra* of S.

We shall need to use the following remark:

Remark 1.1

Let S be a semigroup, and take g to be a function on $S \times S$. For $a, b \in S$, define

$$T_g(\delta_a, \delta_b) = g(a, b),$$

and then extend T_g to be a bilinear function on $\ell_0^1(S) \times \ell_0^1(S)$. In the case where g is bounded by M, T_g extends to a bounded, bilinear functional on $\ell^1(S) \times \ell^1(S)$.

Explicitly,

$$\left| T_g\left(\sum_i \alpha_i \delta_{a_i}, \sum_j \beta_j \delta_{b_j} \right) \right| = \left| \sum_{i,j} \alpha_i \beta_j g(a_i, b_j) \right| \le M \sum_i |\alpha_i| \sum_j |\beta_j|.$$

Now define $\widetilde{T_g} : \ell^1(S) \to \ell^\infty(S)$ by

$$\langle h, \widetilde{T_g}(f) \rangle = T_g(f, h) \quad (f, h \in \ell^1(S)).$$

Then $\widetilde{T_g}$ is a bounded linear map and

$$\langle \delta_b, \widetilde{T_g}(\delta_a) \rangle = g(a, b) \quad (a, b \in S).$$

Throughout the paper, S denotes a countable commutative $nil^\sharp$-semigroup which is the unitization of a nil semigroup S^- (that is, a semigroup S^- with zero such that for all $x \in S^-$, there is an $n \in \mathbb{N}$ such that $x^n = o$), and which is zero-cancellative (that is, for all $a, b, c \in S$, $ab = ac \neq o$ implies $b = c$).

Following [2], we shall write $V_S(x)$ for the set of divisors of x in a unital semigroup S, that is,

$$V_S(x) = \{ y \in S : \exists z \in S, yz = x \}.$$

The set $V_S^*(x)$ is the collection of all functions $g : V_S(x) \to \mathbb{C}$ satisfying the logarithmic condition

$$g(ab) = g(a) + g(b) \quad (ab \in V_S(x)) \qquad \text{(I.1)}$$

In our paper, we shall apply the characterization of cyclic weak amenability of some certain commutative semigroup algebras $\ell^1(S)$, as established in [3], where S is a commutative, 0-cancellative, $nil^\sharp$-semigroup, as introduced in [2] where S is a finite or an infinite semigroup. Then we shall construct some examples where $\ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

II Finite semigroups

In this section we shall discuss cyclic weak amenability of the semigroup algebra $\mathcal{A} = \ell^1(S)$, where S is a finite, commutative, 0-cancellative, $nil^\sharp$-semigroup.

An element $a \neq e \in S$ is an *atom* if $V_S(a) = \{o, e\}$. we shall show that $\mathcal{A}$ is cyclic weakly amenable only when S has exactly one atom, or $S = \{o, e\}$.

Proposition II.1

Let S be a finite, commutative, 0-cancellative, $nil^{\#}$-semigroup with just one atom. Then the semigroup algebra $\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable.

Proof

Suppose that S is a finite, commutative, 0-cancellative, $nil^{\#}$-semigroup with just one atom a. Then the semigroup S can be written as

$T_n = \{e, a, a^2, \dots, a^{n-1}, a^n = o\}$

for some $n \in \mathbb{N}$ with $n \geq 2$. We suppose that $\mathcal{A}_n = \ell^1(T_n)$

Take a derivation $D: \mathcal{A}_n \to \mathcal{A}_n^*$. Then D is bounded because the semigroup algebra $\mathcal{A}_n$ is finite dimensional.

We have $D(\delta_e) = D(\delta_o) = 0$. It can be proved that

$D(\delta_a) = \lambda_e \delta_e^* + \lambda_1 \delta_a^* + \dots + \lambda_{n-2} \delta_{a^{n-2}}^*$

for some $\lambda_e, \lambda_1, \dots, \lambda_{n-2} \in \mathbb{C}$.

We know that D is cyclic if and only if satisfies the equation:

$$\langle f, D(g) \rangle + \langle g, D(f) \rangle = 0 \quad (f, g \in \mathcal{A}). \quad \text{(II.1)}$$

Tak

$f = \delta_{a^k}$ and $g = \delta_a$ for $k = 0, \dots, n-1$, where $a^0 = e$ and $\delta_{a^0} = \delta_e$. Then

$$\langle f, D(g) \rangle + \langle g, D(f) \rangle = \langle \delta_{a^k}, D(\delta_a) \rangle + \langle \delta_a, D(\delta_{a^k}) \rangle$$

$$= \langle \delta_{a^k}, D(\delta_a) \rangle + \langle \delta_a, k\delta_{a^{k-1}} D(\delta_a) \rangle$$

$$= \langle \delta_{a^k}, D(\delta_a) \rangle + k\langle \delta_a, \delta_{a^{k-1}} D(\delta_a) \rangle$$

$$= (k+1)\langle \delta_{a^k}, D(\delta_a) \rangle,$$

and so, by (II.1), we have

$$\langle \delta_{a^k}, \lambda_e \delta_e^* + \lambda_1 \delta_a^* + \dots + \lambda_k \delta_{a^k}^* + \dots + \lambda_{n-2} \delta_{a^{n-2}}^* \rangle = 0,$$

hence $\lambda_k = 0$ for all $k = 0, \dots, n-2$. So $D = 0$. Therefore $\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$.

Thus the proposition is proved. □

Proposition II.2

Let S be a finite, commutative, 0-cancellative, $nil^{\#}$-semigroup with exactly two atoms a and b. Then the semigroup algebra $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable.

Proof

Suppose that the elements a and b are two atoms of the semigroup S. We have to discuss two cases:

1. Let $ab \neq o$. We *claim* that $V_S(ab) = \{e, a, b, ab\}$. For if $u | ab$ then either $u = e$ or there is an atom v such that $v | u$. If $a | u$, then $u = au_1 | ab$ implies that $u_1 | b$, so $u_1 = e$ or $u_1 = b$. Hence ($u = e$ or a or ab). Similarly, if $b | u$, we have $u = b$ or $u = ab$.

Now we define the function $g \in V_S^*(ab)$ by:

$$g(x) = \begin{cases} 0 & if \ x = e \ or \ x = ab \\ 1 & if \ x = a \\ -1 & if \ x = b \end{cases} \quad \text{(II.2)}$$

By using (II.2), it is clear that $g(xy) = g(x) + g(y)$ for all $x, y \in S$ such that $xy | ab$. Thus by [3, Proposition 2.1], the semigroup algebra $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable.

2. Let $ab = o$. Then the semigroup S can be defined as

$$S = \{e, a, a^2, \dots, a^{n-1}, a^n = o = b^m, b, b^2, \dots b^{m-1}, ab = o\},$$

for some $n, m \geq 2$.

A non-zero bounded sensible function $\varphi: C_{(a,b)} \to \mathbb{C}$ can be defined on the class

$$C_{(a,b)} = \{(a,b), (b,a)\}$$

by:

$$\varphi(x,y) = \begin{cases} 1 & if \ x = a \ and \ y = b \\ -1 & if \ x = b \ and \ y = a, \end{cases} \quad \text{(II.3)}$$

So that by [3, Theorem 2.4], the semigroup algebra $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable.

Thus the proposition is proved. □

III. Infinite semigroups

In this section we shall discuss cyclic weak amenability of the semigroup algebra $\mathcal{A} = \ell^1(S)$, where S is an infinite, commutative, 0-cancellative, $nil^{\#}$-semigroup. we shall show that $\mathcal{A}$ is not cyclic weakly amenable if S has at least two distinct atoms.

Proposition III.1

Let S be a commutative, 0-cancellative, $nil^{\#}$-semigroup. Suppose that S has atoms a, b such that $ab = o$. Then the semigroup algebra $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable.

Proof

We shall show that there is a non-zero bounded sensible function φ on the equivalence class $C_{(a,b)} \in M_o^-$ with the equivalence relation defined on the set M_o^- as in [3, Definition 1.3].

We *claim* that

$C_{(a,b)} = \{(a,b),(b,a)\}$. (III.1)

If so, a non-zero bounded sensible function $\varphi: C_{(a,b)} \to \mathbb{C}$ can be defined by:

$$\varphi(x,y) = \begin{cases} 1 & if \;\; x = a \; and \; y = b \\ -1 & if \;\; x = b \; and \; y = a \end{cases},$$

so that by [3, Theorem 2.4], the semigroup algebra $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable.

To prove (III.1), suppose that $(a,b) \sim (p,q)$, so that either we write $a = \alpha\beta$, and $(\alpha\beta, b) \sim (\alpha, \beta b)$. Otherwise we write $b = \gamma\lambda$, and $(a, \gamma\lambda) \sim (a\gamma, \lambda)$. But a is an atom, so if $a = \alpha\beta$ then $\alpha = e$ and $\beta = a$ otherwise if $\alpha = a$ and $\beta = e$ then the new pair is (a,b). Also if $\alpha = e$ and $\beta = a$ then the new pair is $(e, ab) = (e, 0) \notin M_o^-$.

Similarly, since b is an atom, we cannot get a new pair out of $(b\gamma, \lambda)$ with $\gamma\lambda = b$. Thus the proposition is proved. □

Theorem III.2

Let S be a commutative, 0-cancellative, $nil^{\#}$-semigroup. Suppose that S has at least two distinct atoms. Then the semigroup algebra $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable.

Proof

Suppose that S has two distinct atoms a, b.

We have two cases. If $ab = o$, then, by Proposition (III.1), we have that $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable.

Suppose that $ab \neq o$. We claim that there is a non-zero bounded function g in $V_S^*(ab)$ with $g(ab) = 0$. We define g by

$$g(x) = \begin{cases} 1 & if \;\; x = a \\ -1 & if \;\; x = b \\ 0 & otherwise, \end{cases} \quad \text{(III.2)}$$

and we must show that $g(xy) = g(x) + g(y)$ for all $x, y \in V_S(ab)$.

Suppose that $x, y \in V_S(ab)$. We have $g(x) = g(y) = g(xy) = 0$ unless one of x, y, xy is a or b. If $xy = a$ or $xy = b$ then the pair $\{x,y\}$ is $\{e,a\}$ or $\{e,b\}$ and $g(x) + g(y) = g(xy)$ hence we may assume towards a contradiction that $x = a$.

We *claim* that $y = e$ or $y = b$ with $g(x) + g(y) = g(xy)$.

For if y is not e or b we have $ay = xy | ab$ so $y | b$ and b is not an atom. But this is a contradiction. Thus there is $g \in V_S^*(ab)$ with $g(ab) = 0$. Therefore by [3, Theorem 2.4], the semigroup algebra $\mathcal{A} = \ell^1(S)$ is not cyclic weakly amenable. □

IV. Examples

We shall now establish some nice examples for some infinite semigroups, where the semigroup algebra $\ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

Example IV.1

Let S be the semigroup $([0,1], \oplus)$ where

$a \oplus b = \min(a+b, 1) \quad (a, b \in S)$.

Then S is an infinite, commutative, 0-cancellative $nil^{\#}$-semigroup, and the semigroup algebra $\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

Note that, in S, the unit element is 0 and the zero element is 1.

Let $0 < p < 1$. To show that $\mathcal{A}$ is not weakly amenable take the function g to be the identity function, so that $g(x) = x$ for all $x \in V_S(p)$. We have $V_S(p) = [0, p]$, hence $st \in V_S(p)$ implies that $s + t \leq p$, so that

$g(s \oplus t) = s + t = g(s) + g(t)$.

Thus $g \neq 0 \in V_S^*(p)$. By [2, Theorem 1.1], we see that $\mathcal{A}$ is not weakly amenable.

To prove cyclic weak amenability, we have to show that every element h of $V_S^*(p)$ is a multiple of the function g, and then show that there are no bounded sensible functions as before. Since $g(p) \neq 0$, this shows that there is no non-zero element $h \in V_S^*(p)$ with $h(p) = 0$. Let $\alpha = h(p)$. Then the log-condition show that $h(p/n) = \alpha/n$ for all $n \in \mathbb{N}$ and $h(kp/n) = \alpha k/n$ for $k, n \in \mathbb{N}$ and $0 < k \leq n$. We claim that $h(rp) = r\alpha$ for every $0 \leq r \leq 1$ otherwise the function $[0,1] \to \mathbb{R}^+$, $r \mapsto h(rp)$ is discontinuous. Suppose that ρ is irrational with $0 < \rho < 1$ and $h(\rho p) = \rho\beta$ for $\beta \neq \alpha$. Then $h(r\rho p) = r\rho\beta$ for $r \in \mathbb{Q}$ with $r\rho < 1$.

There are rationals $r_n < \rho$ with $r_n \to \rho$ when $n \to \infty$, so that

$h(\rho p) - h(r_n p) = \rho\beta - r_n\alpha \to \rho(\beta - \alpha) \neq 0 \quad as \quad n \to \infty$.

Fix $M > 0$. For large n, we have

$h(\rho p) - h(r_n p) > M(\rho - r_n) > 0$.

But by the log-condition we have

$h((\rho - r_n)p) = h(\rho p) - h(r_n p) > M(\rho - r_n)$.

Take $k \in \mathbb{N}$ with $\frac{1}{2} < k(\rho - r_n) \leq 1$, so that we have ,

$$h(k(\rho - r_n p)) = k(h(\rho p) - h(r_n p))$$
$$> Mk(\rho - r_n) > \frac{1}{2}M,$$

so that the function h is not bounded. But this is a contradiction, because elements of $V_S^*(p)$ must be bounded. Thus $\dim V_S^*(p) = 1$ for $0 < p < 1$ and there is no non-zero function $h \in V_S^*(p)$ with $h(p) = 0$. The function $g_p \in V_S^*(p)$ with $g_p(p) = 1$ is $g_p(x) = x/p$ for all $x \in V_S(p)$.

Now we shall prove that there is no non-zero, bounded, sensible function φ on any equivalence class $C_{(a,b)}$. We first claim that

$C_{(a,b)} = \{(\alpha, \beta) : \alpha + \beta = a + b\}$,

with $a + b \geq 1$.

For say $\alpha < a$, we have $a = \alpha \oplus (a - \alpha)$ so that

$(\alpha \oplus (a - \alpha), b) \sim (\alpha, b + a - \alpha)$.

Similarly for $\beta < b$. But the relation $\sim$ cannot relate pairs (a, b) and (c, d) with $a + b \neq c + d$. So take $\alpha_1, \alpha_2, \beta \in S$ such that $\alpha = \alpha_1 + \alpha_2$ and $\alpha + \beta = a + b$ so that we have

$$\varphi(\alpha_1, \beta + \alpha_2) = g_\alpha(\alpha_1) \cdot \varphi(\alpha, \beta)$$
$$= \frac{\alpha_1}{\alpha} \cdot \varphi(\alpha, \beta), \text{(IV.1)}$$

on the other hand we have

$$\varphi(\alpha, \beta) = -\varphi(\beta, \alpha) = -\varphi(\beta, \alpha_1 + \alpha_2$$
$$= -g_{\alpha_2 + \beta}(\beta) \cdot \varphi(\alpha_2 + \beta, \alpha_1)$$
$$= \beta/(\alpha_2 + \beta) \cdot \varphi(\alpha_1, \alpha_2 + \beta). \quad \text{(IV.2)}$$

If φ is non-zero, choose α, β such that $\varphi(\alpha, \beta) \neq 0$ and by comparing (IV.1) and (IV.2) we must have $\beta/(\alpha_2 + \beta) = \alpha/\alpha_1$ for every

$\alpha = \alpha_1 + \alpha_2$; but the equation $\beta/(\alpha - \alpha_1 + \beta) = \alpha/\alpha_1$ is not true for all values of α_1. Therefore, there is no non-zero bounded sensible function φ, and by [3, Proposition 2.3], the semigroup algebra $\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable.

Example IV.2

Let $A \subset \mathbb{C}$ *be the subset* $\{0\} \cup \{a + \imath b \in \mathbb{C} : 0 < a < 1\}$. *Suppose that* $S = A \cup \{\theta\}$ *such that*

$$s \oplus \theta = \theta \oplus s = \theta \quad ($$

and for each $z, w \in A$ we have

$$z \oplus w = \begin{cases} z + w & if \ z + w \in A \\ \theta & if \ z + w \notin A. \end{cases} \quad \text{(IV.3)}$$

Indeed, the identity of S is 0 and the zero element is θ. Then we claim that S is an infinite, commutative, 0-cancellative $nil^\sharp$-semigroup, and the semigroup algebra

$\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

To show that $\mathcal{A}$ is not weakly amenable take $z \in A$ and define the function g to be the real-part function, so that $g(w) = \Re(w)$ for all $w \in V_S(z)$. We have $V_S(z) = \{w \in S: z \in wS\} = \{a + \iota b: 0 < a < \Re(z)\}$,

(so that the identity function is not bounded on $V_S(z)$). For $u, v \in A$ with $\Re(u) + \Re(v) < 1$ we have $\theta \neq uv \in V_S(z)$ and $g(u \oplus v) = \Re(u) + \Re(v) = g(u) + g(v)$.

Thus $g \neq 0 \in V_S^*(z)$. By [2, Theorem 1.1] we see that $\mathcal{A}$ is not weakly amenable.

Since the identity function is not bounded on the set A, for $z \in A$ the linear space $V_S^*(z)$ does not contain the identity function. We claim that $V_S^*(z)$ is all complex multiples of the function $g(w) = \Re(w) \quad (w \in A)$.

For if we have $w \in V_S(z)$, so that $\Re(w) < \Re(z)$ or $w = z$. We claim that for $g \in V_S^*(z)$ we have

$$g(w) = g(w + \iota\lambda) \quad (w \in V_S(z)).$$

Now for all $w \in V_S(z)$, and for some $\lambda \in \mathbb{R}$, we see that

$$g(\tfrac{w}{2} + \iota\,\tfrac{\lambda}{2}) + g(\tfrac{w}{2} - \iota\,\tfrac{\lambda}{2}) = g(w),$$

also

$$2g(\tfrac{w}{2} + \iota\,\tfrac{\lambda}{2}) = g(w + \iota\lambda) \quad and \quad 2g(\tfrac{w}{2} - \iota\,\tfrac{\lambda}{2}) = g(w - \iota\lambda),$$

so that

$$g(w) = \tfrac{1}{2}(g(w + \iota\lambda) + g(w - \iota\lambda)),$$

or

$$g(w + \iota\lambda) = \tfrac{1}{2}(g(w + 2\iota\lambda) + g(w)),$$

hence

$$g(w + \iota\, 2\lambda) = 2g(w + \iota\lambda) - g(w).$$

Thus

$$g(w + \iota\, 3\lambda) = 2g(w + 2\iota\lambda) - g(w + \iota\lambda) = 3g(w + \iota\lambda) - 2g(w),$$

and similarly for $n \in \mathbb{N}$, we have

$$g(w + \iota\, n\lambda) = ng(w + \iota\lambda) - (n-1)g(w),$$

so that

$$g(w + \iota\, n\lambda) - g(w) = n(g(w + \iota\lambda) - g(w)).$$

By induction, for $n \in \mathbb{N}$ we see that

$$g(w + \iota\, n\lambda) = g(w) + n(g(w + \iota\lambda) - g(w))$$

for all $w \in V_S(z)$,

so we have $g(w + \iota\lambda) = g(w)$ (otherwise g is not bounded). Also we claim that $g(\frac{mz}{n}) = \frac{m}{n} g(z)$ for all $m \leq n$ in $\mathbb{N}$; once again we must have $g(\alpha z) = \alpha g(z)$ for all $0 < \alpha < 1$ otherwise g will not be bounded. Then if $k = \frac{g(z)}{\Re(z)} \in \mathbb{C}$, we have for $0 < \alpha < 1$,

$$g(\alpha z + \iota\lambda) = g(\alpha z) = \alpha g(z) = \alpha k \Re(z) = k\Re(\alpha z + \iota\lambda).$$

Thus $V_S^*(z)$ consists of multiples of the function $g(w) = \Re(w)$ for each $w \in V_S(z)$ and $g_z(w) = \frac{\Re(w)}{\Re(z)}$.

To prove that $\mathcal{A}$ is cyclic weakly amenable we seek to show that there is no non-zero bounded sensible function φ on any equivalence class $C_{(a,b)}$.

We *claim* that

$$C_{(a,b)} = \{(\alpha, \beta): \alpha, \beta \in A \quad and \quad \alpha + \beta = a + b\}.$$

For once cannot have $(a, b) \sim (a', b')$ without $a + b = a' + b'$. Conversely, if $(\alpha, \beta) \in M_o^-$ with $a + b = \alpha + \beta$ we *claim* that $(\alpha, \beta) \in C_{(a,b)}$.

Given $\epsilon > 0$ with $\Re(b) + \epsilon < 1$, and $(a, b) \sim (\Re(a) - \epsilon, \Re(b) + \epsilon)$. Certainly, $(a, b) \sim (b, a)$ so that $C_{(a,b)} = C_{(a-\epsilon, b+\epsilon)}$, so we can assume that $\Re(a) \neq \Re(\alpha)$. If $\Re(a) > \Re(\alpha)$ then $\alpha | a$ in S, $a = \alpha + (a - \alpha)$ for $a, a - \alpha \in A$ so that $(a, b) \sim (\alpha, b + a - \alpha) = (\alpha, \beta)$.

Similarly, if $\Re(a) < \Re(\alpha)$ we have $(a,b) \sim (\alpha,\beta)$ also.

Suppose that φ is a sensible function, so that $\varphi(\frac{a+b}{2},\frac{a+b}{2}) = 0$. If $c + d = a + b$ then either we have $\Re(c) \le \Re(\frac{a+b}{2})$ or $\Re(d) \le \Re(\frac{a+b}{2})$, so we may assume that $\Re(c) \le \Re(\frac{a+b}{2})$ and $\Re(\frac{c}{3}) < \Re(\frac{a+b}{2})$, so $\frac{c}{3} \mid \frac{a+b}{2}$ in S, and

$$\varphi(c,d) = 3\varphi(\frac{c}{3}, d + \frac{2c}{3}) =$$
$$3g_{\frac{a+b}{2}}(\frac{c}{3})\varphi(\frac{a+b}{2},\frac{a+b}{2}) = 0\,.$$

Thus the semigroup algebra $\mathcal{A} = \ell^1(S)$ is cyclic weakly amenable but not weakly amenable.

References

[1] H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs, Volume **24**, Clarendon press, Oxford, 2000.

[2] Hussein M. Ghlaio and Charles J. Read, Irregular abelian semigroups with weakly amenable semigroup algebra, *Semigroup Forum*, **82** (2011), 367-383.

[3] Hussein M. Ghlaio, Cyclic weak amenability of some commutative semigroup algebras, *Journal of Science*, **Vol.**(07) August 2018, 35-38.