TWO CHARACTERIZATIONS OF WEAK AMENABILITY OF COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT

In this paper, we shall describe two nice characterizations of weak amenability of commutative Banach algebras. We shall prove the equivalence between Groenbaek's characterization, and Johnson's characterization for these algebras.

KEYWORDS

Banach algebras, commutative Banach algebras, weak amenability.

1. INTRODUCTION

We follow [1] to recall some definitions and some preliminaries. Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A linear map $D: \mathcal{A} \to X$ is a *derivation* if it satisfies the equation:

$$D(ab) = a \cdot Db + Da \cdot b \quad (a, b \in \mathcal{A})\,.$$

In this paper we shall only consider bounded derivations. Given $x \in X$ and define the map $\delta_x: \mathcal{A} \to X$ by the equation:

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These derivations are *inner* derivations.

Let X^* be the *dual space* of X. Then X^* is a Banach $\mathcal{A}$-bimodule with respect to the operations given by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle \quad \textit{and} \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (a \in \mathcal{A}, x \in X, \lambda \in X^*)\,.$$

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation D from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule X^* is inner, for each Banach $\mathcal{A}$-bimodule X.

A Banach algebra $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule over itself. Then a Banach algebra $\mathcal{A}$ is *weakly amenable* if every bounded derivation $D: \mathcal{A} \to \mathcal{A}^*$ is inner.

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{A}^{\#}$ denotes the Banach algebra formed by adjoining an identity to $\mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra, and suppose that E and F are Banach left and right $\mathcal{A}$-modules, respectively. Then $E \widehat{\otimes} F$ is a Banach $\mathcal{A}$-bimodule for unique products that satisfy

$$a \cdot (x \otimes y) = a \cdot x \otimes y \quad , \quad (x \otimes y) \cdot a = x \otimes y \cdot a \quad (a \in \mathcal{A}, x \in E, y \in F)\,.$$

In particular, $\mathcal{A} \widehat{\otimes} \mathcal{A}^{\#}$ is a Banach $\mathcal{A}$-bimodule.

There is a bounded linear map $\pi: \mathcal{A} \widehat{\otimes} \mathcal{A}^{\#} \to \mathcal{A}$ such that

$$\pi(a \otimes b) = ab \quad (a \in \mathcal{A}, b \in \mathcal{A}^{\#})\,;$$

this map is called *the induced product map*. We see that

$$ker\pi = \left\{\sum_{i=1}^{\infty} a_i \otimes b_i \in \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}^{\#} : \sum_{i=1}^{\infty} a_i b_i = 0\right\} \tag{1.1}$$

Sometimes we shall write $K_{\mathcal{A}}$ for $ker\pi$; it is a closed Banach sub-bimodule in $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}^{\#}$.

Lemma 1.1 *Let $\mathcal{A}$ be a unital Banach algebra. Then*

$$\ker \pi = \overline{\mathrm{lin}}\{a \otimes b - c \otimes d : ab = cd\}\,, \tag{1.2}$$

where, in the right-hand side, $a, c \in \mathcal{A}$ and $b, d \in \mathcal{A}^{\#}$.■

2. Basic algebraic characterization of weak amenability

In his paper [3] Johnson established a characterization for a Banach algebra to be weakly amenable, as follows.

Let $\mathcal{A}$ be a Banach algebra. We define the bounded linear maps

$$D_1 : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{A} \quad \text{and} D_2 : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$$

to be such that

$$D_1(a \otimes b) = ab - ba \quad \text{and} D_2(a \otimes b \otimes c) = a \otimes bc - ab \otimes c + b \otimes ca\,.$$

for all $a, b, c \in \mathcal{A}$. These maps exist by basic properties of the projective tensor product, and clearly $\|D_1\| \leq 2$.

It can be seen that the composition map $D_1 \circ D_2$ is zero, and so $\overline{im}D_2 \subset kerD_1$.

Suppose that D is a bounded linear map from $\mathcal{A}$ to $\mathcal{A}^*$, and define the corresponding linear functional $\mathcal{F}_D$ in $(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^*$ to be such that

$$\mathcal{F}_D(a \otimes b) = \langle b, D(a) \rangle \quad (a, b \in \mathcal{A})\,.$$

Then, for all elements a, b and c in $\mathcal{A}$, we have

$$\begin{aligned}
&(\mathcal{F}_D \circ D_2)(a \otimes b \otimes c) = \mathcal{F}_D(a \otimes bc) - \mathcal{F}_D(ab \otimes c) + \mathcal{F}_D(b \otimes ca) \\
&= \langle bc, D(a) \rangle - \langle c, D(ab) \rangle + \langle ca, D(b) \rangle \\
&= \langle c, D(a) \cdot b \rangle - \langle c, D(ab) \rangle + \langle c, a \cdot D(b) \rangle \\
&= \langle c, D(a) \cdot b - D(ab) + a \cdot D(b) \rangle\,.
\end{aligned}$$

Thus we have that $\mathcal{F}_D \circ D_2 = 0$ if and only if D is a derivation.

Moreover, let $\lambda \in \mathcal{A}^*$. Then the above derivation D is the inner derivation determined by λ if and only if $\mathcal{F}_D = \lambda \circ D_1$. To see this, first set $D = \delta_\lambda$, and take $a, b \in \mathcal{A}$. Then

$$\mathcal{F}_D(a \otimes b) = \langle b, \delta_\lambda(a) \rangle = \langle b, a \cdot \lambda - \lambda \cdot a \rangle = \langle ba - ab, \lambda \rangle\,,$$

and $(\lambda \circ D_1)(a \otimes b) = \langle ba - ab, \lambda \rangle$. Thus $\mathcal{F}_D = \lambda \circ D_1$.

Conversely, suppose that $\mathcal{F}_D = \lambda \circ D_1$. Then

$$\langle b, D(a) \rangle = \langle ab - ba, \lambda \rangle = \langle b, \lambda \cdot a - a \cdot \lambda \rangle \quad (a, b \in \mathcal{A})$$

and so $\mathrm{D}(a) = \lambda \cdot a - a \cdot \lambda = \delta_\lambda(a) \quad (a \in \mathcal{A})$, and hence $D = \delta_\lambda$ is inner.

We write $Z = imD_1$, so that Z is a subspace of $\mathcal{A}$; we also write $\|\cdot\|_\pi$ for the quotient of the projective norm from $\mathcal{A} \widehat{\otimes} \mathcal{A}$ on Z. Thus we see that

$$\|z\| \leq \|D_1\| \|z\|_\pi \leq 2\|z\|_\pi \quad (z \in Z) . \tag{2.1}$$

The following theorem is stated by Johnson in [3]. However, no proof is given in [3], and we do not find the result to be immediate, and so we offer a proof.

Theorem 2.1 *(Johnson) Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Then $\mathcal{A}$ is weakly amenable if and only if $im\ D_1$ is closed and $\overline{im}D_2 = ker\ D_1$ in $\mathcal{A} \widehat{\otimes} \mathcal{A}$.*

Proof Suppose that $\mathcal{A}$ is weakly amenable.We shall first prove that $Z = imD_1$ is closed in $\mathcal{A}$.

Let λ be a continuous linear functional on $(Z, \|\cdot\|_\pi)$, and define

$$\langle b, D_\lambda(a) \rangle = \langle ab - ba, \lambda \rangle \quad (a, b \in \mathcal{A}) . \tag{2.2}$$

Clearly $D_\lambda(a)$ is a linear functional on $\mathcal{A}$, and

$$|\langle b, D_\lambda(a) \rangle| \leq \|\lambda\| \|ab - ba\|_\pi \leq 2\|\lambda\| \|a\| \|b\| \quad (a, b \in \mathcal{A}) , \tag{2.3}$$

so that $D_\lambda(a) \in \mathcal{A}^*$ and $\|D_\lambda(a)\| \leq 2\|\lambda\|\|a\|$ for $a \in \mathcal{A})$, and hence the map $D_\lambda : \mathcal{A} \to \mathcal{A}^*$ is a continuous linear map. It is easily checked that D_λ is a derivation.

Since $\mathcal{A}$ is weakly amenable, there is a continuous linear functional $\mu \in \mathcal{A}^*$ such that

$$D_\lambda(a) = a \cdot \mu - \mu \cdot a \quad (a \in \mathcal{A}) .$$

Clearly, we have

$$\langle ab - ba, \lambda \rangle = \langle ba - ab, \mu \rangle \quad (a, b \in \mathcal{A})$$

and $|\langle z, \mu \rangle| \leq 2\|\mu\| \|z\|_\pi (z \in Z)$ by (2.1) , so $\mu|_Z$ is continuous on $Z, \|\cdot\|_\pi$.

Since λ and μ are continuous linear functionals on $(Z, \|\cdot\|_\pi)$ which agree at elements $ab - ba$ for all $a, b, \in \mathcal{A}$, we have $\lambda = \mu$ on Z . Hence the map $\Theta : \mu \mapsto \mu|_Z$, $\mathcal{A}^* \to (Z, \|\cdot\|_\pi)^*$ is a continuous surjection, and so by the open mapping theorem there is a constant C such that, for each $\lambda \in (Z, \|\cdot\|_\pi)^*$, there exists $\mu \in \mathcal{A}^*$ with $\mu|_Z = \lambda$ and $\|\mu\| \leq C\|\lambda\|$. Thus $\|z\|_\pi \leq C\|z\| \ (z \in Z)$, and so Z is closed in $\mathcal{A}$.

Now, we shall prove that $\overline{im}D_2 = kerD_1$ in $\mathcal{A} \widehat{\otimes} \mathcal{A}$.

Since $D_1 \circ D_2 = 0$, we see that $\overline{im}D_2 \subset kerD_1$, so that it is enough to prove that $kerD_1 \subset \overline{im}D_2$.

Take $\mu \in (\mathcal{A} \widehat{\otimes} \mathcal{A})^*$ with $\mu | \overline{im}D_2 = 0$. Define a map $D : \mathcal{A} \to \mathcal{A}^*$ by

$$\langle b, D(a)\rangle = \langle a \otimes b, \mu\rangle \quad (a, b \in \mathcal{A}) .$$

Then D is a continuous linear map .

We *claim* that D is a derivation. Indeed, for $a, b, c \in \mathcal{A}$, we have

$$\begin{aligned}\langle c, D(ab)\rangle - \langle c, a \cdot D(b)\rangle - \langle c, D(a) \cdot b\rangle &= \langle c, D(ab)\rangle - \langle ca, D(b)\rangle - \langle bc, D(a)\rangle \\ &= \langle ab \otimes c - b \otimes ca - a \otimes bc, \mu\rangle = 0\end{aligned}$$

because $\mu|\overline{im}D_2 = 0$, and so the claim holds.

Since $\mathcal{A}$ is weakly amenable, D is an inner derivation, say $D = \delta_\lambda$ for some $\lambda \in \mathcal{A}^*$. But now

$$\langle a \otimes b, \mu\rangle = (\lambda \circ D_1)(a \otimes b) \quad (a, b \in \mathcal{A}) ,$$

and so $\mu = \lambda \circ D_1$. In particular, $\mu|kerD_1 = 0$. By the Hahn-Banach theorem, $\overline{im}D_2 = kerD_1$.

Conversely, suppose that Z is closed in $\mathcal{A}$ and $\overline{im}D_2 = kerD_1$.
To show that $\mathcal{A}$ is weakly amenable, take a bounded derivation $D: \mathcal{A} \to \mathcal{A}^*$. As above, we have $\mathcal{F}_D \circ D_2 = 0$. Thus $\mathcal{F}_D|\overline{im}D_2 = 0$. But $\overline{im}D_2 = kerD_1$, and so $\mathcal{F}_D|kerD_1 = 0$. This show that $\mathcal{F}_D$ defines a continuous linear functional on $(\mathcal{A} \widehat{\otimes} \mathcal{A})/kerD_1 = imD_1 = (Z, \|\cdot\|_\pi)$. Since Z is closed in $\mathcal{A}$, the norms $\|\cdot\|$ and $\|\cdot\|_\pi$ are equivalent on Z . Hence $\mathcal{F}_D$ is also a continuous linear functional on $(Z, \|\cdot\|)$. By the Hahn–Banach theorem, $\mathcal{F}_D$ extends to $\lambda \in \mathcal{A}^*$. We have $\mathcal{F}_D = \lambda \circ D_1$ on $\mathcal{A} \widehat{\otimes} \mathcal{A}$, and so D is the inner derivation determined by λ . Therefore $\mathcal{A}$ is weakly amenable.Thus the theorem is proved. ■

Note that in the commutative case, we see that $D_1 = 0$, and so $\mathcal{A}$ is weakly amenable if and only if $\overline{im}D_2 = \mathcal{A} \widehat{\otimes} \mathcal{A}$.

In the following theorem, we present the Groenbaek's characterization of weak amenability of commutative Banach algebras. Let $\mathcal{A}$ be a commutative Banach algebra. We recall the subspace $K_\mathcal{A}$ of $\mathcal{A} \widehat{\otimes} \mathcal{A}^\#$ was defined in (1.1).

Theorem 2.2 *(Groenbaek) Let $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ is weakly amenable if and only if $\overline{K_\mathcal{A}^2} = K_\mathcal{A}$.* ■

3. The equivalence of two characterizations

In this section, we shall deal with commutative Banach algebras. We shall prove a nice equivalence between Groenbaek's and Johnson's characterizations; we continue to use same notions and symbols that we mentioned in the previous section. In fact, we restrict to unital algebras.

Theorem 3.1 *Let $\mathcal{A}$ be a unital, commutative Banach algebra. Then the following are equivalent:*
(i)$\mathcal{A}$ is weakly amenable;
(ii)$\overline{K_\mathcal{A}^2} = K_\mathcal{A}$;
(iii)$\overline{\mathrm{im}}D_2 = \mathcal{A} \widehat{\otimes} \mathcal{A}$.

Proof By Theorem 2.2, (i) and (ii) are equivalent. By Theorem 2.1 (in the commutative case), (i) and (iii) are equivalent. We shall give a direct proof that (ii) and (iii) are equivalent, thus giving a proof that is independent of Theorem 2.2 .

We write K for $K_{\mathcal{A}}$, and $e_{\mathcal{A}}$ for the identity of $\mathcal{A}$.

First we suppose that (ii) holds, so that $\overline{K^2} = K$. We shall show that $\overline{im} D_2 = \mathcal{A} \widehat{\otimes} \mathcal{A}$, and hence obtain (iii).

Set $\Lambda = \overline{im} D_2$, so that

$$\Lambda = \overline{lin}\{a \otimes bc - ab \otimes c + b \otimes ca : a, b, c \in \mathcal{A}\} . \tag{3.1}$$

By taking $b = e_{\mathcal{A}}$, we see that $e_{\mathcal{A}} \otimes ca \in \Lambda$ for each $a, b, c \in \mathcal{A}$.

Take $\varphi \in \left(\mathcal{A} \widehat{\otimes} \mathcal{A}\right)^*$ such that $\varphi|_{\Lambda} = 0$.

Let $a, b, c \in \mathcal{A}$. Then

$$\varphi(e_{\mathcal{A}} \otimes ca) = 0 . \tag{3.2}$$

Similarly, putting $c = e_{\mathcal{A}}$ in (3.1), we have $a \otimes b - ab \otimes e_{\mathcal{A}} + b \otimes a$ in Λ , and so

$$\varphi(a \otimes b) + \varphi(b \otimes a) = \varphi(ab \otimes e_{\mathcal{A}}) . \tag{3.3}$$

Now we have

$$(e_{\mathcal{A}} \otimes a - a \otimes e_{\mathcal{A}})(e_{\mathcal{A}} \otimes b - b \otimes e_{\mathcal{A}}) = e_{\mathcal{A}} \otimes ab + ab \otimes e_{\mathcal{A}} - a \otimes b - b \otimes a ,$$

so that, from (3.2) and (3.3), we have

$$\begin{aligned}\varphi((e_{\mathcal{A}} \otimes a - a \otimes e_{\mathcal{A}})(e_{\mathcal{A}} \otimes b - b \otimes e_{\mathcal{A}})) &= \varphi(e_{\mathcal{A}} \otimes ab) + \varphi(ab \otimes e_{\mathcal{A}}) \\ &-\varphi(a \otimes b) - \varphi(b \otimes a) = 0 .\end{aligned}$$

We now *claim* that $\varphi(\overline{K^2}) = 0$. By Lemma 1.1, it is sufficient to show that

$$\varphi((a \otimes b - a' \otimes b')(u \otimes v - u' \otimes v')) = 0$$

whenever $a, a', u, u', b, b', v, v' \in \mathcal{A}$ with $ab = a'b' = m_1$ and $uv = u'v' = m_2$. Then $(a \otimes b - a' \otimes b')(u \otimes v - u' \otimes v')$ is an element in K^2 , and, by using the following equalities:

$$\varphi(au \otimes bv) = \varphi(a \otimes ubv) + \varphi(u \otimes bva) = \varphi(a \otimes m_2 b) + \varphi(u \otimes m_1 v) ;$$

$$\varphi(a'u \otimes b'v) = \varphi(a' \otimes ub'v) + \varphi(u \otimes b'va') = \varphi(a' \otimes m_2 b') + \varphi(u \otimes m_1 v) ;$$

$$\varphi(au' \otimes bv') = \varphi(a \otimes u'bv') + \varphi(u' \otimes bv'a) = \varphi(a \otimes m_2 b) + \varphi(u' \otimes m_1 v') ;$$

$$\varphi(a'u' \otimes b'v') = \varphi(a' \otimes u'b'v') + \varphi(u' \otimes b'v'a') = \varphi(a' \otimes m_2 b') + \varphi(u' \otimes m_1 v') ;$$

we see that

$$\begin{aligned}\varphi((a \otimes b - a' \otimes b')(u \otimes v - u' \otimes v')) &= \varphi(au \otimes bv) - \varphi(a'u \otimes b'v) \\ &-\varphi(au' \otimes bv') + \varphi(a'u' \otimes b'v') = 0 .\end{aligned}$$

Thus $\varphi(\overline{K^2}) = 0$ as required. It follows from (ii), that $\varphi|_K = 0$ because $\overline{K^2} = K$.
Now take $z = \sum_{i=1}^{\infty} a_i \otimes b_i \in \mathcal{A} \widehat{\otimes} \mathcal{A}$, with $\sum_{i=1}^{\infty} \|a_i\|\|b_i\| < \infty$. Then

$$z = \sum_{i=1}^{\infty} (a_i \otimes b_i - e_{\mathcal{A}} \otimes a_i b_i) + \sum_{i=1}^{\infty} e_{\mathcal{A}} \otimes a_i b_i \in K + \Lambda ,$$

because, for each $i \in \mathbb{N}$, we have $a_i \otimes b_i - e_{\mathcal{A}} \otimes a_i b_i \in K$, and $e_{\mathcal{A}} \otimes a_i b_i \in \Lambda$, as we remarked, and because both Λ and K are closed. So $\varphi(z) = 0$. Hence $\varphi = 0$. By the Hahn–Banach theorem, $\Lambda = \mathcal{A} \widehat{\otimes} \mathcal{A}$, as required.

We now suppose that (iii) holds, so that $\overline{imD_2} = \mathcal{A} \widehat{\otimes} \mathcal{A}$. We need to prove that $\overline{K^2} = K$.

Let $a, b, c \in \mathcal{A}$. Then

$$\begin{aligned} a \otimes bc - ab \otimes c + b \otimes ca &= e_{\mathcal{A}} \otimes abc - (e_{\mathcal{A}} \otimes a - a \otimes e_{\mathcal{A}})(e_{\mathcal{A}} \otimes b - b \otimes e_{\mathcal{A}}) e_{\mathcal{A}} \otimes c \\ &\in e_{\mathcal{A}} \otimes abc + K^2 . \end{aligned}$$

Now take $x \in K$, so that $\pi(x) = 0$. Since $\overline{imD_2} = \mathcal{A} \widehat{\otimes} \mathcal{A}$, we can write $x = \lim_{n\to\infty} r_n$, say, where $r_n \in imD_2 \quad (n \in \mathbb{N})$. We have $\pi(r_n) \to 0$ as $n \to \infty$.
By the definition of D_2 , we may write

$$r_n = \sum_{i=1}^{\infty} \left(a_{n,i} \otimes b_{n,i} c_{n,i} + b_{n,i} \otimes a_{n,i} c_{n,i} - a_{n,i} b_{n,i} \otimes c_{n,i}\right) ,$$

where $\sum_{i=1}^{\infty} \|a_{n,i}\|\|b_{n,i}\|\|c_{n,i}\| < \infty$. Thus we have

$$\begin{aligned} r_n &= \sum_{i=1}^{\infty} \left(e_{\mathcal{A}} \otimes a_{n,i} b_{n,i} c_{n,i} + (e_{\mathcal{A}} \otimes a_{n,i} - a_{n,i} \otimes e_{\mathcal{A}})(e_{\mathcal{A}} \otimes b_{n,i} - b_{n,i} \otimes e_{\mathcal{A}}) e_{\mathcal{A}} \otimes c_{n,i}\right) \\ &= e_{\mathcal{A}} \otimes \pi(r_n) + \sum_{i=1}^{\infty} \left((e_{\mathcal{A}} \otimes a_{n,i} - a_{n,i} \otimes e_{\mathcal{A}})(e_{\mathcal{A}} \otimes b_{n,i} c_{n,i} - b_{n,i} \otimes c_{n,i}\right) \end{aligned}$$

because $\pi(r_n) = \sum_{i=1}^{\infty} a_{n,i} b_{n,i} c_{n,i}$ and because $\mathcal{A}$ is commutative.
Thus we have $r_n \in e_{\mathcal{A}} \otimes \pi(r_n) + \overline{K^2}$.

But $\pi(r_n)$ converges to zero as $n \to \infty$, so

$$x = \lim_{n\to\infty} r_n = \lim_{n\to\infty} (r_n - e_{\mathcal{A}} \otimes \pi(r_n)) \in \overline{K^2} .$$

Therefore $K \subset \overline{K^2}$, and so (ii) follows.
Thus the theorem is proved. ■

In fact, by a small extra argument, the result holds even when $\mathcal{A}$ does not have an identity.

REFERENCES

1. H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs, Volume **24**, Clarendon press, Oxford, 2000.
2. N. Groenbaek, A characterization of weakly amenable Banach algebras, *Studia Math.*, **94** (1989), 149-161.
3. B. E. Johnson, Derivations from $L^1(G)$ into $L^1(G)$ and $L^\infty(G)$, *Porc. of the Harmonic Analysis Conference, Lecture Notes in Maths*, (1988), 191-198.