Order Projection of Linear Operators

Manal Altaher Elzidani[1], Zieneb Ali Elshegmani[2]
[1,2]Misurata University, Faculty of Education
Misurata Libya

m.elzidani@edu.misuratau.edu.ly[1] z.elshegmani@edu.misuratau.edu.ly[2]

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Abstract

Short abstract. *Order (or band) projections are the most important subjects in functional analysis and its applications. This paper studies a special class of positive linear operator known as order projections, and provides some of its application which are; extension theorem for linear operators, theory of order continuous operators, and the components of positive operators. A useful comparison property of order projection is described.*

I. INTRODUCTION

In linear algebra and functional analysis a projection is a linear transformation P from a vector space to itself for example $P^2 = P$ whenever P is applied twice to any value, it gives the same result as if it were applied once (idempotent). Many researchers studied and analyzed the order projections and their related subjects, some of them; [Schott 2013] studied an iterative method to solve linear operator equations sequences of linear iterative operator occur which have a nontrivial projection kernel. [Laura 2013] devoted the set of all products with an orthogonal projection and a positive operator, and related the factorization with the notion of compatibility and explored the polar decomposition of the operator. [Corach 2012] they extend the relationship between closed unbounded idempotents and dense decomposition of a Hillbert space to notion of compatibility between closed subspaces and positive bounded operators.

In this research we shall study a special class of positive operators known as order (or band) projections, first we will review a few properties of order dense Riesz subspaces, and study important theorems describe the basic properties of order dense ideals. In fact, the band generated by any set coincides with the band generated by the ideal generated by the same set, it was also clarified that not every band is a projection band. Finally, the basic properties of order projections are summarized in the important theorem.

II. CONCEPTS AND THE STUDY

Definition 1 [Kreyszing 1978]
An element x in an ordered vector space E is called positive whenever $x \geq 0$.
The set of all positive elements of E will be denoted by E^+.

Definition 2 [Kreyszing 1978]
Recall that a Riesz subspace G of a Riesz space E is said to be **Order dense** in E whenever for each $0 < x \in E$, there exists some $y \in G$ with $0 < y \leq x$.

Definition 3 [Kreyszing 1978]
A Riesz space E is called **Archimedean** whenever $n^{-1}x \downarrow 0$ holds in E for each $x \in E^+$.
The following characterization of order dense Riesz subspaces in Archimedean Riesz spaces will be used freely in this research.

Theorem 1 A Riesz subspace G of an Archimedean Riesz space E is order dense in E if and only if $\{y \in G : 0 \leq y \leq x\} \uparrow x$ hold for each $x \in E^+$.

Proof.
If $\sup x = \{y \in G : 0 \leq y \leq x\}$ hold for each $x \in E^+$, then G is clearly order dense in E.
For the converse, assume that G is order dense in E, and let $x \in E^+$. Assume by way of contradiction that some $z \in E$ satisfies $z < x$ and $y \leq z$ for each $y \in G$ with $0 < y \leq x$.

Then, by the order denseness of G in E there exists some $u \in G$ with $0 < u \leq x - z$. From $0 < u \leq x$ we see that $u \leq z.$ and so

$$0 < 2u = u + u \leq x - z + z = x.$$

By induction, $0 < nu \leq x$ hold for each $n,$ contradicting the Archimedean property of *E* Thus,

$$\{y \in G : 0 \leq y \leq x\} \uparrow x$$

holds in $E,$ and the proof of the theorem is finished.

Consider an order dense Riesz subspace G of $E.$ It is useful to know that the embedding of G into E preserves arbitrary suprema and infima. This result is stated next.

Theorem 2 [Carl 1990]
Let G be either an ideal or an order dense Riesz subspace of a Riesz space $E,$ and let $D \subseteq G^+$ satisfy $D \downarrow$. Then $D \downarrow 0$ holds in G if and only if $D \downarrow 0$ holds in $E.$

Definition 4 [Kreyszing 1978]
Recall that a subset A of a Riesz space is called **solid** whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A.$

Definition 5 [Charalambos 1985]
A solid vector subspace is referred to as an **ideal**.

It's obvious that if A and B are solid subsets of a Riesz space, then their algebraic sum is

$$A + B = \{a + b : a \in A \, and \, b \in B\}$$

Is likewise a solid set. In particular, the algebraic sum of two ideal is also an ideal.

Definition 6 [Charalambos 1985]
disjoint complement A^d is defined by

$$A^d = \{x \in E : x \perp y \; for \, all \, y \in A\}.$$

We write A^{dd} for $(A^d)^d$. Not that

$$A \cap A^d = \{0\}.$$

Next theorem describes the basic properties of order dense ideals. Keep in mind that the disjoint complement of an arbitrary nonempty set of Riesz space is always ideal.

Theorem 3
For an ideal *A* of a Riesz space *E* the following statements hold:

1. The ideal A is order dense in E if and only if $A^d = \{0\}$.
2. The ideal $A \oplus A^d$ is order dense in *E*.
3. The ideal A is order dense in A^{dd}.

Proof.
(1) Let A be order dense in E, and let $x \in A^d$. If $x \neq 0$ holds, then there exists some $y \in A$ with $0 < y < |x|$.

This implies $y \in A \cap A^d = \{0\},$ a contradiction. Thus, $A^d = \{0\}$ holds.

For the converse, assume that $A^d = \{0\}$ holds, and let $0 < x \in E$. If $y \wedge x = 0$ holds for all $x \in A^+$, then $x \in A^d = \{0\}$ must also hold. Thus, $y \wedge x > 0$ holds for some $x \in A^+.$

Then $y \wedge x \in A$ and $0 < y \wedge x \leq 0$ show that A is order dense in E.

(2) If $x \perp A \oplus A^d$, then $x \perp A$ and $x \perp A^d$ both hold.
Therefore, $x \in A^d \cap A^{dd} = \{0\},$ which shows that $(A \oplus A^d)^d = \{0\}$. By (1) $A \oplus A^d$ is order dense in E.

(3) This follows immediately from (1).

Definition 7 [Kreyszing 1978]

In Riesz space a net $\{x_\alpha\}$ is said to be **order convergent** to x (in symbols $x_\alpha \xrightarrow{0} x$) whenever there exists another net $\{y_\alpha\}$ (with the same indexed set) satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all α (abbreviated as $|x_\alpha - x| \leq y_\alpha \downarrow 0$

Definition 8 [Kreyszing 1978]
A subset*A* of a Riesz space is said to be **order closed** whenever $\{x_\alpha\} \subseteq A$ and $x_\alpha \xrightarrow{0} x$ imply $x \in A.$

Not that a solid subset A is order closed if and only if $\{x_\alpha\} \subseteq A$ and $0 < x_\alpha \uparrow x$ imply $x \in A$. Indeed, if the solid set A has the latter property and a net $\{x_\alpha\} \subseteq A$ satisfies $x_\alpha \xrightarrow{0} x$, then there $|x_\alpha|$ and $0 \leq (|x| - y_\alpha)^+ \uparrow |x|$, we easily see that $x \in A.$

Definition 9 [John 1990]
An order closed ideal is referred to as a **band.**
Thus, by the above discussion an ideal A is a band if and only if $\{x_\alpha\} \subseteq A$ and $0 < x_\alpha \uparrow x$ imply $x \in A$ (or, equivalently, if and only if $D \subseteq A^+$ and $D \uparrow x$ imply $x \in A$).

In the early development of Riesz spaces a band was called a normal subspace.

Let *A*be a nonempty subset of a Riesz space *E*. Then **the ideal generated** by *A* is the smallest (with respect to inclusion) ideal that contains *A*.

A moment is thought reveals that this ideal is

$$\left\{\begin{matrix} x \in E : \exists x_1, \dots, x_n \in A and \\ \lambda_1, \dots, \lambda_n \in R^+ \; with |x| \le \sum_{i=1}^{n} \lambda_i |x_i| \end{matrix}\right\}.$$

The ideal generated by an element *x* will be denoted by A_x. By the above

$$A_x = \{ y \in E : \exists\, \lambda > 0 \; with \; |y| \le \lambda |x| \}.$$

Every ideal of the form A_x is referred to as a **principal ideal.** Similarly, **the band generated** by A is the smallest band that contains A. Such a band always exists (since it is the intersection of the family of all bands that contain A, and E is one of them).
Clearly, the band generated by *A* coincides with the band generated by the ideal generated by *A*.
The band generated by an ideal is described as follows.

Theorem 4
If *A* is an ideal of a Riesz space *E*, then the band generated by *A* is precisely

$$\{ x \in E : \exists\, \{x_\alpha\} \subseteq A^+ \; with \; 0 \le x_\alpha \uparrow |x| \}.$$

In particular, every ideal is order dense in the band it generates. Moreover, the band *B*, generated by a single element x satisfies

$$B_x = \{ y \in E : |y| \wedge n|x| \uparrow |y| \}.$$

Proof.
Let

$$B = \{ x \in E : \exists\, \{x_\alpha\} \subseteq A^+ \; with \; 0 \le x_\alpha \uparrow |x| \} \quad .$$

Clearly every band containing A must contain *B*. Thus, to establish our result it is enough to show that *B* is a band.

To this end, let $x, y \in B.$ Pick two nets $\{x_\alpha\} \subseteq A^+$ and $\{y_\beta\} \subseteq A^+$ with $0 \le x_\alpha \uparrow |x|$ and $0 \le y_\alpha \uparrow |y|.$
From

$$|x+y| \wedge (x_\alpha + y_\beta) \uparrow |x+y| \wedge (|x| + |y|) = |x+y|$$

And

$$|\lambda| \; x_\alpha \uparrow |\lambda\, x|$$

We see that *B* is vector subspace. Also, if $|z| \le |x|$ holds, then from

$$\{ |z| \wedge x_\alpha \} \subseteq A \text{ and } 0 \le |z| \wedge x_\alpha \uparrow |z| \wedge |x| = |z|,$$

It follows that $z \in B$. Hence, B is an ideal. Finally, to see that B is a band, let $\{x_\alpha\} \subseteq B$ satisfy $0 \le x_\alpha \uparrow |x|$.

Put $D = \{ y \in A : \exists some\, \alpha \; with 0 \le y \le x_\alpha \}$.

Then $D \subseteq A^+$ and $D \uparrow x$ hold. Therefore, $x \in B$, and so B is a band.
To establish the formula for B_x, let $y \in B_x$.
By the above there exists a net $\{x_\alpha\} \subseteq A_x$ with $0 \le x_\alpha \uparrow |y|$. Now given α there exists some n with $x_\alpha \le n|x|$, and so $x_\alpha \le |y| \wedge n|x| \le |y|$ holds. This easily implies $|y| \wedge n|x| \uparrow |y|$, and our conclusion follows.

It is obvious that A^d is always a band. It is important to know that the band generated by a set A is precisely A^{dd} .

Theorem 5
The band generated by a nonempty subset A of an Archimedean Riesz space is precisely A^{dd} (and hence if A is a band, then $A = A^{dd}$ holds).
Proof.
We have said before that the band generated by *A* is the same as the band generated by the ideal generated by *A*. Therefore we can assume that A is an ideal. By theorem 3 we see that A is order dense in A^{dd}, and hence (by Theorem 1) for each $x \in A^{dd}$ there exists a net $\{x_\alpha\} \subseteq A$ with $0 \le x_\alpha \uparrow |x|$.
This easily implies that A^{dd} is the smallest band containing *A*.
A useful condition under which an ideal is necessarily a band is presented next

Theorem 6
Let *A* and *B* be two ideal in a Riesz space *E* such that $E = A \oplus B$. Then *A* and *B* are both bands satisfying $A = B^d$ and $B = A^d$ (and hence $A = A^{dd}$ and $B = B^{dd}$ both hold).
Proof:
Note first that for each $a \in A$ and $b \in B$ we have

$$|a| \wedge |b| \in A \cap B = \{0\},$$

And so $A \perp B$. In particular, $A \subseteq B^d$.
On the other hand, if $x \in B^d$, then write $x = a + b$ with $a \in A,$ $b \in B,$ and note that $b = x - a \in B \cap B^d = \{0\}$ implies $x = a \in A.$
Thus, $B^d \subseteq A$, and so $A = B^d$ holds. This shows that A is a band. By the symmetry of the situation $B = A^d$ also hold.

Definition 10 [Charalambos 1985]
A band B in a Riesz space E that satisfies $E = B \oplus B^d$ is referred to as a **projection band.**

The next result characterizes the ideals that are projection bands.

Theorem 7

For an ideal B in a Riesz space E the following statements are equivalent:

1. B is a projection band, $E = B \oplus B^d$ holds.
2. For each $x \in E^+$ the supremum of the set $B^+ \cap [0, x]$ exists in E and belongs to *B*.
3. There exists an ideal *A* of *E* such that $E = B \oplus A$ holds.

Proof:

(1) $\Rightarrow$ (2) Let $x \in E^+$, choose the (unique) elements $0 \le y \in B$, and $0 \le z \in B^d$ with $x = y + z$. If $u \in B^+$ satisfies $u \le x = y + z$, then it follows from $0 \le (u - y)^+ \le z \in B^d$ and $(u - y)^+ \in B$ that $(u - y)^+ = 0$. Thus, $u \le y$, and so y is an upper bound of the set $B^+ \cap [0, x]$. Since $y \in B \cap [0, x]$, we see that $y = \sup\{u \in B^+ : u \le x\} = \sup B \cap [0, x]$ holds in *E*.

(2$\Rightarrow$(3)fix some $x \in E^+$, and let $u = \sup B \cap [0, x]$. Clearly, u belongs to B. Put $y = x - u \ge 0$. If $0 \le w \in B$, then $0 \le y \wedge w \in B$, and moreover from $0 \le u + y \wedge w \in B$
and
$u + y \wedge w = (u + y) \wedge (u + w) = x \wedge (u + w) \le x$,
it follows that
$u + y \wedge w \le u$. Hence, $y \wedge w = 0$ holds, and so $y \in B^d$. From $x = u + y$, we see that $E = B \oplus B^d$, and therefore (3) holds with $A = B^d$.

(3)$\Rightarrow$(1) this follows from theorem 6.

Not every band is a projection band, and a Riesz space in which every band is projection band is referred to as a Riesz space with the **projection property.** From the preceding theorem it should be clear that in a Dedekind complete Riesz space every band is a projection band. This was proven by F. Riesz [Charalambos 1985] in one of his early fundamental papers on Riesz spaces. Because it guarantees an abundance of order projections, we state it next as a separate theorem.

Theorem 8

If *B* is a band in a Dedekind complete Riesz space *E*, then $E = B \oplus B^d$ holds.

As usual, an operator*P* on a vector space is called a **Projection** whenever $P = P^2$. If a projection P is defined on a Riesz space and P is also a positive operator, then P will be referred to as **a positive Projection**.

Now let *B* be a projection band in a Riesz space *E*. Thus, $E = B \oplus B^d$ holds, and so every element $x \in E$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. Then it is easy to see that a projection $P_B : E \to E$ is defined by

$$P_B(x) = x_1 .$$

Clearly P_B is a positive projection. Any projection of the form P_B is called an **order projection** (or a **band projection**). Thus, the order projection are associated with the projection bands in a one-to-one fashion.

Theorem 9

If B is a projection band in a Riesz space *E*, then

$$P_B(x) = \sup\{y \in B : 0 \le y \le x\},$$

holds for all $x \in E^+$.

Proof:

Let $x \in E^+$, then (by theorem 7)

$u = \sup\{y \in B : 0 \le y \le x\}$ exists and belongs to *B*.

We claim that $u = P_B(x)$.

Write $x = x_1 + x_2$ with $0 \le x_1 \in B$ and $0 \le x_2 \in B^d$, and note that $0 \le x_1 \le x$ implies $0 \le x_1 \le u$. Thus, $0 \le u - x_1 \le x - x_1 = x_2$, and hence $u - x_1 \in B^d$ since $u - x_1 \in B$ and $B \cap B^d = \{0\}$, we see that $u = x_1$, as claimed.

Among all projections the order projections are characterized as follows.

Theorem 10

For an operator $T : E \to E$ on a Riesz space the following statements are equivalent:

1. *T*is order projection.
2. *T*is a projection satisfying $0 \le T \le I$ (where, of course, *I* is the identity operator on *E*).
3. *T* and*I-T* have disjoint ranges, that is, $Tx \perp y - Ty$ holds for all $x, y \in E$.

Proof:

(1) $\Rightarrow$ (2) obvious.

(2 $\Rightarrow$ (3) Let $x, y \in E^+$, put $z = Tx \wedge (I - T)y$. From the inequality $z \le (I - T)y$ it follows that $0 \le Tz \le (I - T)y = (T - T^2)y = 0$, and so $Tz = 0$. Similarly, $(I - T)z = 0$, and hence $z = (I - T)z + Tz = 0$ holds.

This shows that T and $I - T$ have disjoint ranges.

(3) $\Rightarrow$ (1) Let A and B be the ideals generated by the ranges of T and $I - T$, respectively.

By our hypothesis it follows that $A \perp B$, and from $x = Tx + (I - T)x$ we see that $E = A \oplus B$.
Hence, by theorem 6 both A and B are projection bands of *E*. Now the identity

$$P_A x - Tx = P_A x - P_A Tx = P_A(x - Tx) = 0$$

shows that $T = P_A$ holds, thus T is an order projection, and the proof of the theorem is finished.

From the previous we noted that a positive projection need not be an order projection, as the next example shows.

Example1 [John B 1990]

consider the operator $T : L_1[0,1] \to L_1[0,1]$ defined by

$$T(f) = \left(\int_0^1 f(x)dx \right) \cdot \qquad (1)$$

where Eq.**(1)** denotes the constant function one, clearly $0 \le T = T^2$ holds, and it is not difficult to see that T is not an order projection.
The basic properties of order projections are summarized in the next theorem.

Theorem 11

If *A* and *B* are projection bands in Riesz space *E*, then A^d, $A \cap B$, and $A + B$ are likewise projection bands. Moreover, they satisfy

1. $P_{A^d} = I - P_A$;
2. $P_{A \cap B} = P_A P_B = P_B P_A$; and
3. $P_{A+B} = P_A + P_B - P_A P_B$.

Proof

(1) From $E = A \oplus A^d$ it follows that $A^{dd} = A$ holds (see theorem 6), and so A^d is a projection band. The identity $P_{A^d} = I - P_A$ should be obvious.

(2) To see that $A \cap B$ is a projection band note that the identity $B \cap [0,x] = [0, P_B x]$ implies $A \cap B \cap [0,x] = A \cap [0, P_B x]$ for each $x \in E^+$.
Thus,

$$P_A P_B x = \sup\ A \cap B \cap [0,x] = \sup A \cap [0, P_B x]$$

holds for each $x \in E^+$, which (by theorem 7) shows that $A \cap B$ is a projection band and that $P_{A \cap B} = P_A P_B$ holds. Similarly, $P_{A \cap B} = P_A P_B$.

(3) Assume at the beginning that the two projection bands A and B satisfy $A \perp B$. Let $x \in E^+$. If $0 \le a + b \in A + B$ satisfy $a + b \le x$, then clearly $a \in A \cap [0,x]$ and $b \in B \cap [0,x]$, and so $a + b \le P_A x + P_B x \in A + B$ holds.

This shows that

$$\sup(A + B) \cap [0,x] = P_A x + P_B x \in A + B,$$

And hence by theorem7 the ideal $A + B$ is a projection band. Also, $P_{A+B} = P_A + P_B$ holds.

Now the general case can be established by observing that $A + B = A \cap B^d + B$. In addition, we have

$$P_{A+B} = P_{A \cap B^d + B} = P_{A \cap B^d} + P_B = P_A P_{B^d} + P_B$$
$$= P_A(I - P_B) + P_B = P_A - P_A P_B + P_B,$$
$$= P_A + P_B - P_{A \cap B}$$

so the proof of the theorem is finished.

An immediate consequence of statement (2) of the preceding theorem is that two arbitrary order projection mutually commute. A useful comparison property of order projection is described next.

Theorem 12

If *A* and *B*are projection bands in Riesz space *E*, then the following statements are equivalent:

1. $A \subseteq B$
2. $P_A P_B = P_B P_A = P_A$; and
3. $P_A \le P_B$.

Proof:

(1) $\Rightarrow$ (2) Let $A \subseteq B$,then from theorem1 it follows that

$$P_A P_B = P_B P_A = P_{A \cap B} = P_A$$

(2) $\Rightarrow$ (3) For each $0 \le x$ we have $P_A x = P_B P_A x \le P_B x$, and so $P_A \le P_B$ holds.

(3)$\Rightarrow$(1) If $0 \le x \in A$, then it follows from

$$0 \le x = P_A x \le P_B x \in B$$

that $x \in B$. Therefore, $A \subseteq B$ holds, as required.

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