STABILITY OF FUZZY LOTKA -VOLTIRA MODEL USING LIAPUNOV FUNCTION

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Abstract

The Liapunov function is one of the method that using to test the stability of the solutions of the differential equations. In this paper we will use this approach to study the stability of the solution of the uncertainty population models. Lotka-Voltirra model with fuzzy initial conditions has chosen to explore the uncertainty in population models. The fuzzy and deterministic stages of the model will be compared and numerical example will be provided.

Keywords: Liapunov function; Fuzzy Lotka-Voltirra model; stability; Uncertainty.

1. Introduction

The study of the Stability of population dynamics is a very important concept. In deterministic models, the stability of the steady state solutions becomes an important subject, by examining what happens in a steady state which gives us better understand the behavior of a system. The predator-prey model is one of the most popular in mathematical ecology. The dynamics properties of this model have significant biological background which represent the basis of models used today in the analysis of population dynamics. In our real life, the most real word phenomenon and physical experiments cannot be described fully 100%, partly because the full information basically is not available which introduce the uncertainty in the models. The probability theory is one of the tools that used to treat the uncertainty in population models. The effect of randomness investigated by (Soong, 1973), (Tu & Wilman, 1992), (Kegan & West, 2005), (Gonzaléz, Jódar, Villanueva, & Santonja, 2008), (Pollett, Dooley, & Ross, 2010) and (Omar & Abu-Hasan, 2010, 2011, 2012; Omar & Hasan, 2013). The authors adopt different random distributions and different approaches to introduce the random uncertainty in the models that describe those real world phenomenon. The fuzzy set theory is another strong tool to treat the uncertainty in the models when this uncertainty is considered as a result of unclear and lake information. In the literature, (Xu & Gertner, 2009) introduced an uncertainty analysis technique, the general Fourier Amplitude Sensitivity Test (FAST), to study uncertainties in transient population dynamics. They found that the general FAST is able to identify the amount of uncertainty in transient dynamics and contributions by different demographic parameters. They applied the general FAST to a mountain goat (Oreamnos americanus) matrix population model to give a clear illustration of how uncertainty analysis can be conducted for transient dynamics arising from matrix population models. (Normah Maan, 2013) used fuzzy delay differential equations to discuss on the theory and analysis of delay predator-prey model with consider the uncertainty in the parameters. (Akın & Oruç, 2012) considered a prey-predator model with fuzzy initial values. They use the concept of generalized differentiability and obtain graphical solutions for the problem under consideration. They adopted analytical approach to solve the fuzzy prey predator model, which gives the estimate of number of prey and number of predator at time t. (Peixoto, Barros, Bassanezi, & Fernandes, 2015) described the interaction between the prey, Aphis glycines (Hemiptera: Aphididae) - the soybean aphid, and its predator, Orius insidiosus (Hemiptera: Anthocoridae). The investigation was to develop a simple and specific methodology by fuzzy rule-based system to help enhance decision making tools for biological control of this pest. (da Silva Peixoto, de Barros, & Bassanezi, 2008) studied the fuzzy predator-prey population model. They elaborated the classical deterministic model by means of a fuzzy rule-based system. They also studied the stability of the critical points of the Holling–Tanner model. (Omar, Ahmed, & Hasan, 2015) studied the predator-prey model with ratio dependent functional response to explore the uncertainty in population models. It was done by assuming triangular fuzzy number as the initial states of the model.

In this paper, uncertainty is considered in the basic Lotka-Voltirra model of the case where the initial conditions are triangular fuzzy numbers. The main goal of this paper is to study the stability of the fuzzy behavior of the model using the Liapunov function and compare the fuzzy solution with the deterministic.

2. Preliminary and Concepts

A fuzzy number is a convex fuzzy set $\widetilde{A}$ of $\mathbb{R}$, for which the following conditions hold:

i. $\widetilde{A}$ is normalized. i.e. $\exists\, x \in \mathbb{R} : \mu_{\widetilde{A}\,(x)} = 1$, where $\mu_{\widetilde{A}\,(x)}$ is the membership function of a fuzzy number $\widetilde{A}$.

ii. $\mu_{\widetilde{A}\,(x)}$ is upper semicontinuous.

iii. $\{x \in \mathbb{R} : \mu_{\widetilde{A}\,(x)} = r\}$ are compact sets for $0 < r \leq 1$.

We say that a fuzzy number is Trapezoidal (Triangular, Gaussian) fuzzy number if its membership function is Trapezoid (Triangular, Gaussian). The membership function of a Trapezoidal fuzzy number will be interpreted as follows:

$$\mu_{\widetilde{A}\,(x)} = \begin{cases} 0 & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 \leq x \leq a_4 \\ 0 & x > a_4 \end{cases} \qquad (1)$$

If $F(\mathbb{R})$ is the set of all fuzzy numbers, and $\widetilde{A} \in F(\mathbb{R})$, we can characterize $\widetilde{A}$ by its $r-$cuts by following closed bounded intervals:

$$[\widetilde{A}\,]_r = \{x \in \mathbb{R} : \mu_{\widetilde{A}\,(x)} \geq r\} = [a_1^r, a_2^r], \quad 0 < r \leq 1 \qquad (2)$$

$$[\widetilde{A}\,]_0 = \overline{\{x \in \mathbb{R} : \mu_{\widetilde{A}\,(x)} \geq r\}} = \overline{[a_1^r, a_2^r]}, \quad 0 < r \leq 1 \qquad (3)$$

Operations on fuzzy numbers can be described as follows:

If $\widetilde{A}, \widetilde{B} \in F(\mathbb{R})$, then for $0 < r \leq 1$

1. $[\widetilde{A} + \widetilde{B}]_r = [a_1^r + b_1^r, a_2^r + b_2^r]$,
2. $[\widetilde{A} - \widetilde{B}]_r = [a_1^r - b_1^r, a_2^r - b_2^r]$,
3. $[\widetilde{A}.\widetilde{B}]_r = [\min\{a_1^r b_1^r, a_1^r b_2^r, a_2^r b_1^r, a_2^r b_2^r\}, \max\{a_1^r b_1^r, a_1^r b_2^r, a_2^r b_1^r, a_2^r b_2^r\}]$,
4. $[s\widetilde{A}\,]_r = s[\widetilde{A}\,]_r$ where s is scalar.

Let $d_H : F(\mathbb{R}) \times F(\mathbb{R}) \to \mathbb{R}_+ \cup \{0\}$, $d_H(\widetilde{A}, \widetilde{B}) = \mathrm{Sup}_{0 \leq r \leq 1} \max\{|a_1^r - b_1^r|, |a_2^r - b_2^r|\}$ be Hausdorff distance between fuzzy numbers, and $(F(\mathbb{R}), d_H)$

iv. Definition 1:

Let I be a real interval. A mapping $X: I \to F(\mathbb{R})$ is called a fuzzy process. We denote to fuzzy process as:

$$[X(t)]_r = [x(t)_1^r, x(t)_2^r], \quad t \in I \text{ and } 0 < r \leq 1. \tag{4}$$

The fuzzy derivative of a fuzzy process x(t) as in Sikkala [1] is defined by:

$$[X'(t)]_r = \left[{x'}_1^r(t), {x'}_2^r(t)\right], \quad t \in I \text{ and } 0 < r \leq 1. \tag{5}$$

v. 2.1 Fuzzy initial fuzzy problem (FIVP)

The initial value problem can consider as

$$\frac{dx(t)}{dt} = f\big(t, x(t)\big) \;;\; x(0) = X_0\,, \tag{6}$$

Where $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping and $X_0 \in F(\mathbb{R})$ with r −cut intervals

$$[X_0]_r = [x_{01}^r, x_{02}^r], \qquad 0 < r \leq 1. \tag{7}$$

When $X = X(t)$ is a fuzzy number, the extension principle of Zadeh leads to the following definition

$$f\big(t, x(\tau)\big)(s) = \sup\{x(\tau): s = f(t, \tau)\}\ ,\ s \in \mathbb{R}, \tag{8}$$

It follows that

$$[f(t, x(t))]_r = [f_1^r(t, x(t)), f_2^r(t, x(t))], \quad 0 < r \leq 1. \tag{9}$$

where

$$f_1^r\big(t, x(t)\big) = \min\{f(t, w) : \ w \in [x_1^r(t), x_2^r(t)]\}, \quad 0 < r \leq 1, \tag{10}$$

$$f_2^r\big(t, x(t)\big) = \min\{f(t, w) : \ w \in [x_1^r(t), x_2^r(t)]\}, \quad 0 < r \leq 1. \tag{11}$$

vi. 3.1 Liapunov Function

Let V(x) be a scalar function of the components of x. By $\dot{V}(x)$, the derivative of V along solutions of system of an autonomous system of differential equations

$$x' = f(x) \tag{12}$$

We mean

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x) \tag{13}$$

$\dot{V}(x)$ represents how V changes (increases or decreases along solution).

$V(x)$ is said to be a positive definite function if $V(0) = 0$, and outside the origin, but inside some domain D containing the origin, $V(x) > 0$.

Suppose now that system (1) has an equilibrium at $x = 0$, i.e., $f(0) = 0$ (if $f(x_0) = 0$, $x_0 \neq 0$, the change of variables $y = x - x_0$ gives a new system with an equilibrium at the origin). The following results give stability criteria for $x = 0$ of system (1).

Let $V(x)$ be a positive definite function in D. Let $\Omega \subset D$ be a sub region of D containing the origin in its interior.

1. $\dot{V}(x) \leq 0,\ x \in \Omega$, then $x = 0$ is stable.
2. $\dot{V}(x) < 0,\ x \in \Omega$, then $x = 0$ is asymptotically stable and the region of attraction contains Ω.
3. $\dot{V}(x) > 0, x \in \Omega$, then $x = 0$ is unstable.

In the case that $\dot{V}(x) \leq 0$, $\dot{V}(x)$ is called a Liapunov Function. Also if $\dot{V}(x) = 0$, $x \in \Omega$, then the origin is a center, i.e., all solutions lying in Ω are periodic solutions, and their equations in the phase plane are given by the family $V(x) = c$.

3. Lotka-Volterra system

We assume that the prey growth, if left alone, is Malthusian, and the specific growth rate is diminished by an amount proportional to the predator density

$$\frac{dx}{dt} = x(\alpha - \beta y) \qquad \alpha, \beta > 0 \tag{14.a}$$

For the predator species it assumed that in the absence of prey, the predators will become extinct exponentially but their growth rate is enhanced by an amount proportional to the prey density

$$\frac{dy}{dt} = y(-\gamma + \delta x) \qquad \gamma, \delta > 0 \tag{14.b}$$

System (14) is that system referred to as Lotka-Volterra model for predator-prey interactions where $x, y \geq 0$ (Murray, 2002).

vii. 3.1 Solution of the Lotika-Volterra model:

We consider again system (14)

$$\frac{dx}{dt} = x(\alpha - \beta y) \tag{14.a}$$

$$\frac{dy}{dt} = y(-\gamma + \delta x) \tag{14.b}$$

Then

$$\frac{dy}{dx}=\frac{y(-\gamma+\delta x)}{x(\alpha-\beta y)} \tag{15}$$

Or

$$\frac{\alpha-\beta y}{y}dy=\frac{-\gamma+\delta x}{x}dx \tag{16}$$

Suppose we initiate our system at $x_0, y_0 > 0$ then

$$\int_{y_0}^{y}(\frac{\alpha}{v}-\beta)\,dv=\int_{x_0}^{x}\left(\frac{-\gamma}{u}+\delta\right)du \tag{17}$$

Or

$$\alpha\ln\left(\frac{y}{y_0}\right)-\beta(y-y_0)=-\gamma\ln\left(\frac{x}{x_0}\right)+\delta(x-x_0) \tag{18}$$

In order to examine Eq. (18) properly, we first write it as

$$\alpha\ln\left(\frac{\beta y/\alpha}{\beta y_0/\alpha}\right)-\beta\left(\left(y-\frac{\alpha}{\beta}\right)-(y_0-\frac{\alpha}{\beta})\right)$$
$$=-\gamma\ln\left(\frac{\delta x/\gamma}{\delta x_0/\gamma}\right)+\delta\left(\left(x-\frac{\delta}{\gamma}\right)-(x_0-\frac{\delta}{\gamma})\right) \tag{19}$$

Or

$$\delta\left(x-\frac{\delta}{\gamma}\right)-\gamma\ln\left(\frac{\delta x}{\gamma}\right)+\beta\left(y-\frac{\alpha}{\beta}\right)-\alpha\ln\left(\frac{\beta y}{\alpha}\right)$$
$$=c_0 \tag{20}$$

Where

$$c_0=\delta\left(x_0-\frac{\delta}{\gamma}\right)-\gamma\ln\left(\frac{\delta x_0}{\gamma}\right)+\beta\left(y_0-\frac{\alpha}{\beta}\right)$$
$$-\alpha\ln\left(\frac{\beta y_0}{\alpha}\right) \tag{21}$$

Now, let

$$m_1=x-\frac{\gamma}{\delta}\ ,\qquad m_2=y-\frac{\alpha}{\beta} \tag{22}$$

Which when substituted into (20) gives

$$\delta m_1-\gamma\ln\left(\frac{\delta m_1}{\gamma}+1\right)+\beta m_2-\alpha\ln\left(\frac{\beta m_2}{\alpha}+1\right)$$
$$=c_0 \tag{23}$$

At this point we let

$$\Psi_1(m_1)=\delta m_1-\gamma\ln\left(\frac{\delta m_1}{\gamma}+1\right) \tag{24.a}$$

$$\Psi_2(m_2) = \beta m_2 - \alpha \ln\left(\frac{\beta m_2}{\alpha} + 1\right) \qquad (24.b)$$

And note that

$$\frac{d\Psi_1(m_1)}{dm_1} = \frac{\delta^2 m_1}{\delta m_1 + \gamma} \quad \text{and} \quad \frac{d\Psi_2(m_2)}{dm_2} = \frac{\beta^2 m_2}{\beta m_2 + \alpha} \qquad (25)$$

From (24), $\Psi_i(0) = 0$, $i = 1,2$; from (25), $m_i\left(\frac{d\Psi_i(m_i)}{dm_i}\right) > 0$, m_i , $i = 1,2$, so long as $m_1 > -\frac{\gamma}{\delta}$ and $m_2 > -\frac{\alpha}{\beta}$, which by (22) is equivalent to $x, y > 0$. Further, $c_0 > 0$ since

$$c_0 = \Psi_1\left(x_0 - \frac{\gamma}{\delta}\right) + \Psi_2\left(y_0 - \frac{\alpha}{\beta}\right) \qquad (26)$$

This means that Eq. (20) which be written as

$$\Psi_1(m_1) + \Psi_2(m_2) = c_0 \qquad (27)$$

satisfies the criteria of Liapunove functions for the existence of a periodic solution for arbitrary $x_0, y_0 > 0$, and so all solutions initiating in the first quadrant are periodic.

This includes as a special case when $x_0 = \frac{\gamma}{\delta}$, $y_0 = \frac{\alpha}{\beta}$, in which case $c_0 = 0$, and hence corresponds to an equilibrium.

It only remains to examine the solutions initiating on the positive axes. If $x_0 > 0$, $y_0 = 0$, then $y \equiv 0$ and x satisfies

$$\frac{dx}{dt} = \alpha x \qquad x(0) = x_0 \qquad (28)$$

Which we have already solved, giving

$$x = x_0\, e^{\alpha t} \qquad (29)$$

Hence the x −axis is itself a solution with the flow away from the origin.

If $y_0 > 0$, $x_0 = 0$, then $x \equiv 0$ and y satisfies

$$\frac{dy}{dt} = -\gamma y \qquad y(0) = y_0 \qquad (30)$$

Here the solution is

$$y = y_0\, e^{-\gamma t} \qquad (31)$$

Which means the y −axis is also a solution with the flow toward the origin. The origin itself is, of course, an equilibrium, and it is a hyperbolic point.

4. System of fuzzy initial conditions

In system (14) when the initial conditions are fuzzy numbers becomes

$$\frac{dX}{dt} = X(\alpha - \beta Y) \qquad X(0) = X_0, \quad \alpha, \beta > 0 \qquad (32.a)$$

$\frac{dY}{dt} = Y(-\gamma + \delta X) \qquad Y(0) = Y_0, \qquad \gamma, \delta > 0 \qquad (32.b)$

Where $X_0, Y_0 > 0$ are fuzzy numbers, their $r-$cut defined as

$$[X_0]_r = [x_{01}^r, x_{02}^r], \qquad 0 < r \leq 1. \qquad (33)$$

$$[Y_0]_r = [y_{01}^r, y_{02}^r], \qquad 0 < r \leq 1. \qquad (34)$$

The fuzzy processes $X, Y: I \to F(\mathbb{R})$ are defined as

$$[X(t)]_r = [x(t)_1^r, x(t)_2^r], \quad t \in I \text{ and } 0 < r \leq 1. \qquad (35)$$

$$[Y(t)]_r = [y(t)_1^r, y(t)_2^r], \quad t \in I \text{ and } 0 < r \leq 1. \qquad (36)$$

Using Sikkala derivative for $0 < r \leq 1$ we rewrite the system (32)

$$\frac{d[X(t)]_r}{dt} = F([X(t)]_r, [Y(t)]_r, \alpha, \beta) \qquad X(0) = X_0, \quad \alpha, \beta > 0 \qquad (37.a)$$

$$\frac{dY(t)]_r}{dt} = G([X(t)]_r, [Y(t)]_r, \gamma, \delta) \qquad X(0) = Y_0, \quad \gamma, \delta > 0 \qquad (37.b)$$

where

$$F([X(t)]_r, [Y(t)]_r, \alpha, \beta) = [X(t)]_r(\alpha - \beta[Y(t)]_r) \qquad (38.a)$$

$$G([X(t)]_r, [Y(t)]_r, \alpha, \beta) = [Y(t)]_r(-\gamma + \delta[X(t)]_r) \qquad (38.b)$$

and

$$x'^r_1 = \min F([X(t)]_r, [Y(t)]_r, \alpha, \beta) \qquad (39.a)$$

$$x'^r_2 = \max F([X(t)]_r, [Y(t)]_r, \alpha, \beta) \qquad (39.b)$$

$$y'^r_1 = \min G([X(t)]_r, [Y(t)]_r, \gamma, \delta) \qquad (40.c)$$

$$y'^r_2 = \max G([X(t)]_r, [Y(t)]_r, \gamma, \delta) \qquad (40.d)$$

Now for $0 < r \leq 1$ we will start to solve the fuzzy system (36)

$$\frac{[dY(t)]_r}{[dx(t)]_r} = \frac{G([X(t)]_r, [Y(t)]_r, \gamma, \delta)}{F([X(t)]_r, [Y(t)]_r, \alpha, \beta)} \qquad (41)$$

$$\frac{[dy_1^r, dy_2^r]}{[dx_1^r, dx_2^r]} = \frac{[\min G([X(t)]_r, [Y(t)]_r, \gamma, \delta), \max G([X(t)]_r, [Y(t)]_r, \gamma, \delta)]}{[\min F([X(t)]_r, [Y(t)]_r, \alpha, \beta), \max F([X(t)]_r, [Y(t)]_r, \alpha, \beta)]} \qquad (42)$$

$$\left[\min_{\substack{i=1,2\\j=1,2}} \left\{\frac{dy_j^r}{dx_i^r}\right\}, \max_{\substack{i=1,2\\j=1,2}} \left\{\frac{dy_j^r}{dx_i^r}\right\}\right] = \frac{\left[\min_{\substack{i=1,2\\j=1,2}} \left\{y_i^r\left(-\gamma + \delta x_j^r\right)\right\}, \max_{\substack{i=1,2\\j=1,2}} \left\{y_i^r\left(-\gamma + \delta x_j^r\right)\right\}\right]}{\left[\min_{\substack{i=1,2\\j=1,2}} \left\{x_i^r\left(\alpha - \beta y_j^r\right)\right\}, \max_{\substack{i=1,2\\j=1,2}} \left\{x_i^r\left(\alpha - \beta y_j^r\right)\right\}\right]} \qquad (43)$$

Now for fixed $k, l \in i$ and fixed $p, q \in j$ we get

$$\left[\frac{dy_p^r}{dx_k^r},\frac{dy_q^r}{dx_l^r}\right] = \left[\frac{y_p^r(-\gamma+\delta x_k^r)}{x_k^r(\alpha-\beta y_p^r)},\frac{y_q^r(-\gamma+\delta x_l^r)}{x_l^r(\alpha-\beta y_q^r)}\right] \quad (44)$$

First we will discuss

$$\frac{dy_p^r}{dx_k^r}=\frac{y_p^r(-\gamma+\delta x_k^r)}{x_k^r(\alpha-\beta y_p^r)} \quad (45)$$

Which can write it as

$$\frac{(\alpha-\beta y_p^r)}{y_p^r}dy_p^r=\frac{(-\gamma+\delta x_k^r)}{x_k^r}\,dx_k^r, \quad (46)$$

Since $[x_0]_r\,,[y_0]_r>0$ then $x_{01}^r\,,y_{01}^r>0$ for $0<r\le 1$ we get

$$\int_{y_{01}^r}^{y_p^r}\left(\frac{\alpha}{v}-\beta\right)dv=\int_{x_{01}^r}^{x_k^r}\left(\frac{-\gamma}{u}+\delta\right)du, \quad (47)$$

Or

$$\alpha\ln\left(\frac{y_p^r}{y_{01}^r}\right)-\beta\left(y_p^r-y_{01}^r\right) = -\gamma\ln\left(\frac{x_k^r}{x_{01}^r}\right)+\delta(x_k^r-x_{01}^r) \quad (48)$$

In order to examine Eq. (48) properly, we first write it as

$$\alpha\ln\left(\frac{\beta y_p^r/\alpha}{\beta y_{01}^r/\alpha}\right)-\beta\left(\left(y_p^r-\frac{\alpha}{\beta}\right)-(y_{01}^r-\frac{\alpha}{\beta})\right) = -\gamma\ln\left(\frac{\delta x_k^r/\gamma}{\delta x_{01}^r/\gamma}\right) + \delta\left(\left(x_k^r-\frac{\delta}{\gamma}\right)-(x_{01}^r-\frac{\delta}{\gamma})\right) \quad (49)$$

Or

$$\delta\left(x_k^r-\frac{\delta}{\gamma}\right)-\gamma\ln\left(\frac{\delta x_k^r}{\gamma}\right)+\beta\left(y_p^r-\frac{\alpha}{\beta}\right)-\alpha\ln\left(\frac{\beta y_p^r}{\alpha}\right) = c_0 \quad (50)$$

Where

$$c_0 = \delta\left(x_{01}^r - \frac{\delta}{\gamma}\right) - \gamma \ln\left(\frac{\delta x_{01}^r}{\gamma}\right) + \beta\left(y_{01}^r - \frac{\alpha}{\beta}\right) - \alpha \ln\left(\frac{\beta y_{01}^r}{\alpha}\right) \qquad (51)$$

Now, let

$$m_{1,r} = x_k^r - \frac{\gamma}{\delta} \quad , \qquad m_{2,r} = y_p^r - \frac{\alpha}{\beta} \qquad (52)$$

Which when substituted into (50) gives

$$\delta m_{1,r} - \gamma \ln\left(\frac{\delta m_{1,r}}{\gamma} + 1\right) + \beta m_{2,r} - \alpha \ln\left(\frac{\beta m_{2,r}}{\alpha} + 1\right) = c_0 \qquad (53)$$

At this point we let

$$\Psi_{1,r}(m_{1,r}) = \delta m_{1,r} - \gamma \ln\left(\frac{\delta m_{1,r}}{\gamma} + 1\right) \qquad (54.a)$$

$$\Psi_{2,r}(m_{2,r}) = \beta m_{2,r} - \alpha \ln\left(\frac{\beta m_{2,r}}{\alpha} + 1\right) \qquad (54.b)$$

And note that

$$\frac{d\Psi_{1,r}(m_{1,r})}{dm_{1,r}} = \frac{\delta^2 m_{1,r}}{\delta m_{1,r} + \gamma} \quad , \qquad \frac{d\Psi_{2,r}(m_{2,r})}{dm_{2,r}} = \frac{\beta^2 m_{2,r}}{\beta m_{2,r} + \alpha} \qquad (55)$$

From (54), $\Psi_{i,r}(0) = 0$, $i = 1,2$; from (55), $m_{i,r}\left(\frac{d\Psi_{i,r}(m_{i,r})}{dm_{i,r}}\right) > 0$, $m_{i,r}$, $i = 1,2$, so long as $m_{1,r} > -\frac{\gamma}{\delta}$ and $m_{2,r} > -\frac{\alpha}{\beta}$, which by (52) is equivalent to $x_k^r, y_p^r > 0$. Further, $c_0 > 0$ since

$$c_0 = \Psi_{1,r}\left(x_{01}^r - \frac{\gamma}{\delta}\right) + \Psi_{2,r}\left(y_{01}^r - \frac{\alpha}{\beta}\right) \qquad (65)$$

this means that Eq. (50) which be written as

$$\Psi_{1,r}(m_{1,r}) + \Psi_{2,r}(m_{2,r}) = c_0 \qquad (66)$$

satisfies the criteria of Liapunove Functions for the existence of a periodic solution for arbitrary $x_{01}^r, y_{01}^r > 0$, and so all solutions initiating in the first quadrant are periodic.

This includes as a special case when $x_{01}^r = \frac{\gamma}{\delta}$, $y_{01}^r = \frac{\alpha}{\beta}$, in which case $c_0 = 0$, and hence corresponds to the steady state.

It only remains to examine the solutions initiating on the positive axes. If $x_{01}^r > 0$, $y_{01}^r = 0$, then $y_p^r \equiv 0$ and x_k^r satisfies

$$\frac{dx_k^r}{dt} = \alpha x \qquad x_k^r(0) = x_{01}^r \tag{67}$$

Which we have already solved, giving

$$x_k^r = x_{01}^r e^{\alpha t} \tag{68}$$

Hence the x −axis is itself a solution with the flow away from the origin.

If $y_{01}^r > 0$, $x_{01}^r = 0$, then $x_k^r \equiv 0$ and y satisfies

$$\frac{dy_p^r}{dt} = -\gamma y_p^r \qquad y_p^r(0) = y_{01}^r \tag{69}$$

Here the solution is

$$y_p^r = y_{01}^r \, e^{-\gamma t} \tag{70}$$

Which means the y −axis is also a solution with the flow toward the origin. The origin itself is, of course, an equilibrium, and it is a hyperbolic point.

The same results we can get with substitute x_{02}^r by x_{01}^r and y_{02}^r by y_{01}^r

$$\frac{dy_q^r}{dx_l^r} = \frac{y_q^r(-\gamma + \delta x_l^r)}{x_l^r\left(\alpha - \beta y_q^r\right)} \tag{71}$$

As a result, for arbitrary fuzzy initial conditions

$$[x_0]_r = [x_{01}^r, x_{02}^r] > [0,0] = 0 \tag{72.a}$$

$$[y_0]_r = [y_{01}^r, y_{02}^r] > [0,0] = 0 \tag{72.b}$$

and for all $0 < r \leq 1$, all fuzzy solutions $[x(t)\,]_r$ and $[y(t)\,]_r$ initiating in the first quadrant are periodic.

This includes as a special case when $[x_{01}^r, x_{02}^r] = [\frac{\gamma}{\delta}\ , \frac{\gamma}{\delta}] = \frac{\gamma}{\delta}$, $[y_{01}^r, y_{02}^r] = \left[\frac{\alpha}{\beta}\ , \frac{\alpha}{\beta}\right] = \frac{\alpha}{\beta}$, in which case $c_0 = 0$, and hence corresponds to an equilibrium.

If $[x_{01}^r, x_{02}^r] > 0 = [0\,,0]$ and $[y_{01}^r, y_{02}^r] = [0,0]$, then $[y_1^r, y_2^r] \equiv [0,0]$and $[x_1^r, x_2^r]$ satisfies

$$\frac{d[x_1^r, x_2^r]}{dt} = \alpha[x_1^r, x_2^r] \qquad X(0) = [x_1^r(0), x_2^r(0)] = [x_{01}^r, x_{02}^r] \tag{73}$$

Which we have already solved, giving

$$[x_1^r, x_2^r] = [x_{01}^r, x_{02}^r]e^{\alpha t} \tag{74}$$

Hence the x −axis is itself a fuzzy solution with the flow away from the origin.

If $[y_{01}^r, y_{02}^r] > [0,0] = 0$, $[x_{01}^r, x_{02}^r] = 0$, then $[x_1^r, x_2^r] \equiv 0$ and $[y_1^r, y_2^r]$ satisfies

$$\frac{d[y_1^r, y_2^r]}{dt} = -\gamma[y_1^r, y_2^r] \qquad Y(0) = [y_1^r(0), y_2^r(0)] = [y_{01}^r, y_{02}^r] \qquad (75)$$

Here the fuzzy solution is

$$[y_1^r, y_2^r] = [y_{01}^r, y_{02}^r]\, e^{-\gamma t} \qquad (76)$$

which means the $y-$axis is also a fuzzy solution with the flow toward the origin. The origin itself is, of course, an equilibrium, and it is a hyperbolic point.

5. Numerical Example

Consider the fuzzy lotka-Volterra model

$$\frac{dX}{dt} = X(\alpha - \beta Y) \qquad X(0) = \widetilde{X}_0, \qquad (77.a)$$

$$\frac{dY}{dt} = Y(-\gamma + \delta X) \qquad Y(0) = \widetilde{Y}_0, \qquad (77.b)$$

Here: $\alpha = 0.21$, $\beta = 0.44$, $\gamma = 0.22$, $\delta = 0.27$ and the fuzzy initial conditions are defined as

$$[\widetilde{X}_0]^r = [6 + 2r, 10 - 2r] \quad , \quad [\widetilde{Y}_0]^r = [0.5 + 1.5r, 3.5 - 1.5r] \qquad (78)$$

To solve the system (77), he dependency problem should be taken into account. The method that introduced by (Alsonosi Omar & Abu Hasan, 2012) is useful to compute the fuzzy behavior of the predator and its prey. Fig. 1 shows the crisp behavior and the interaction between the species. The criteria of Liapunov are satisfied and all the solutions lying in the first quadratic are periodic.

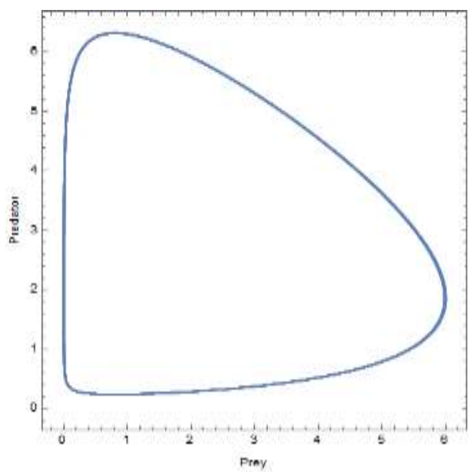

Figure 1. The crisp behavior and the phase plane of the predator and its prey.

The fuzzy behavior is shown in Fig. 2. From the system (77) and the fuzzy initial conditions (78), It can be clearly seen that the criteria of Liapunov is also satisfied which means that the fuzzy behavior lying in the first quadratic is also periodic (see Fig. 3.)

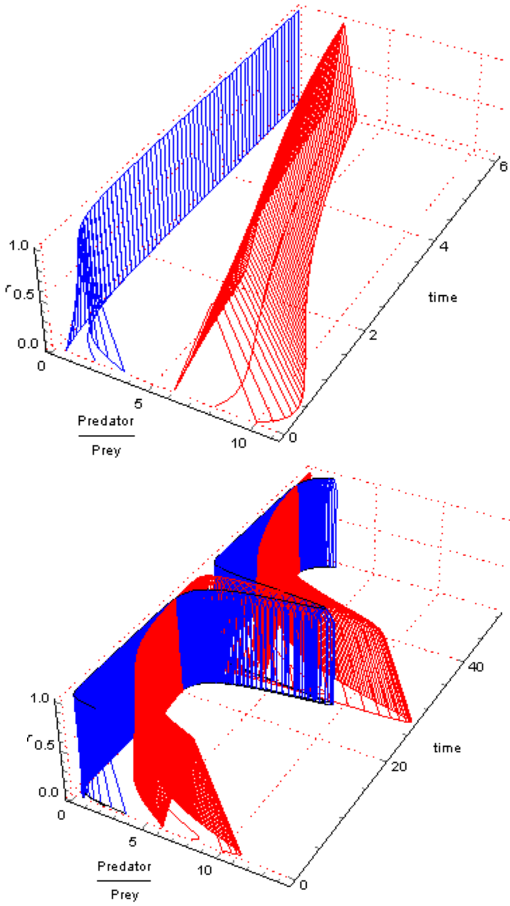

Figure 2. The fuzzy behavior of the predator and its prey over time.

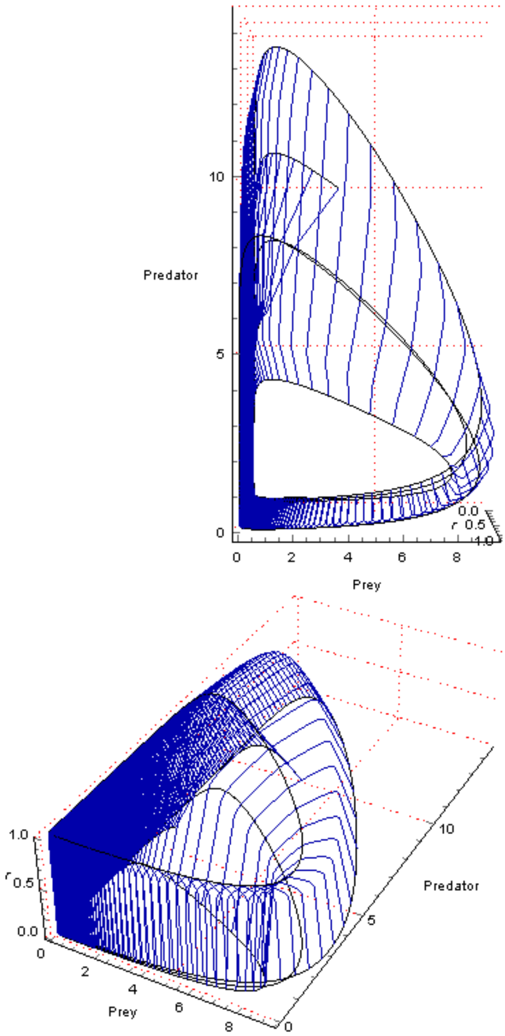

Figure 3. The fuzzy interaction between the predator and its prey over time.

In general, the fuzzy behavior represents the generalization of the crisp behavior and makes the description of the phenomenon more realistic than the classical one. Moreover, the interactions between the components of the models will not be classical. It will take the form of uncertainty that is limited by the fuzzy range, which is different for every possibility degree.

6. Conclusion

It can be concluded that the comparison between the deterministic and the fuzzy Lotka-Voltira model was developed. Lipunove function is used to make comparison between the crisp and the fuzzy behavior of the predator prey model. The fuzziness made the generalization on the crisp behavior. That means, when the classical predator prey model satisfied the criteria of Lipunove, the fuzzy model is conducted. Then, the fuzzy behavior of predator prey model can be periodic.

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